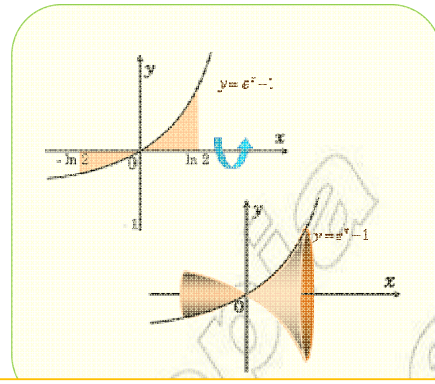
Unit 5



INTRODUCTION TO INTEGRAL CALCULUS

Unit Outcomes:

After completing this unit, you should be able to:

- *understand the concept of definite integral.*
- *integrate polynomial functions, simple trigonometric functions, exponential and logarithmic functions.*
- *use the various techniques of integration to evaluate a given integral.*
- *use the fundamental theorem of calculus for computing definite integrals.*
- *apply the knowledge of integral calculus to solve real life mathematical problems.*

Main Contents

INTRODUCTION

YOU HAVE JUST SEEN differential calculus, WHICH IS ONE OF THE TWO BRANCHES OF CALCULUS. IN THIS UNIT YOU SHALL SEE THE OTHER BRANCH OF CALCULUS, CALLED integral calculus. INTEGRATION IS THE REVERSE PROCESS OF DIFFERENTIATION. IT IS THE PROCESS OF FI… FUNCTION ITSELF WHEN ITS DERIVATIVE IS KNOWN.

FOR EXAMPLE, IF THE SLOPE OF A TANGENT AT AN ARBITRARY POINT OF A CURVE IS KNOWN… POSSIBLE TO DETERMINE THE EQUATION OF THE CURVE USING THE METHOD OF INTEGRAL CA… IT IS POSSIBLE TO FIND DISTANCE OF A MOVING OBJECT IN TERMS OF TIME, IF ITS VELO… ACCELERATION IS KNOWN.

DIFFERENTIAL CALCULUS DEALS WITH RATE OF CHANGE OF FUNCTIONS, WHEREAS INTEG… DEALS WITH TOTAL SIZE OR VALUE SUCH AS AREAS ENCLOSED BY CURVES, VOLUMES OF … LENGTHS OF A CURVES, TOTAL MASS, TOTAL FORCE, ETC.

DIFFERENTIAL CALCULUS AND INTEGRAL CALCULUS ARE CONNECTED BY A THEOREM … fundamental theorem of calculus.

IN INTEGRAL CALCULUS THERE ARE TWO KINDS OF INTEGRATIONS WHICH ARE CALLED TH… indefinite integral OR THE ANTI DERIVATIVE AND THE definite integral.

THE INDEFINITE INTEGRAL OR THE ANTI DERIVATIVE INVOLVES FINDING THE FUNCTIO… DERIVATIVE IS KNOWN.

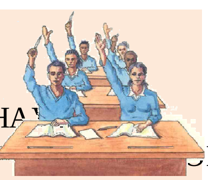
THE DEFINITE INTEGRAL, DENOTED BY $\int_a^b f(x)\,dx$ IS INFORMALLY DEFINED TO BE THE SIGNED AREA OF THE REGION IN THE xy-PLANE BOUNDED BY THE CURVE $y = f(x)$, THE x-AXIS AND THE VERTICAL LINES $x = a$ AND $x = b$.

ONE OF THE MAIN GOALS OF THIS UNIT IS TO EXAMINE THE THEORY OF INTEGRAL CALC… INTRODUCE YOU TO ITS NUMEROUS APPLICATIONS IN SCIENCE AND ENGINEERING.

5.1 INTEGRATION AS REVERSE PROCESS OF DIFFERENTIATION

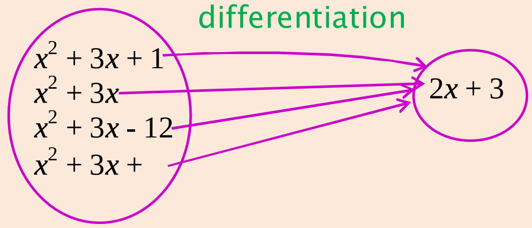
5.1.1 The Concept of Indefinite Integral

ACTIVITY 5.1



1. FIND AT LEAST THREE DIFFERENT FUNCTIONS WHICH HAVE DERIVATIVE $2x$. DESCRIBE SIMILARITIES (AND DIFFERENCES) BETWEEN … FUNCTIONS YOU FOUND.
2. WRITE THE SET OF ALL FUNCTIONS WITH DERIVATIVE $2x$.

3 CHECK THAT ALL OF THE FUNCTIONS:
$f(x) = x^2 + 3x + 1$, $g(x) = x^2 + 3x$, $h(x) = x^2 + 3x - 12$ AND $k(x) = x^2 + 3x +$
HAVE THE SAME DERIVATIVE $2x + 3$.

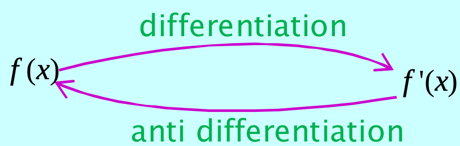


IN GENERAL, $\frac{d}{dx}\left(x^2 + 3x + c\right) = 2x + 3$ FOR ANY CONSTANT c.

DRAW THE GRAPHS OF f, g, h AND k TOGETHER, USING SAME PAIR OF AXES OF REFERENCE.

Definition 5.1

THE PROCESS OF FINDING $f(x)$ FROM ITS DERIVATIVE $f'(x)$ IS SAID TO BE ANTI DIFFERENTIATION OR INTEGRATION. $f(x)$ IS SAID TO BE THE **anti derivative** OF $f'(x)$.



Integration is the reverse operation of differentiation.

Definition 5.2

THE SET OF ALL ANTI DERIVATIVES OF A FUNCTION $f(x)$ IS CALLED THE **indefinite integral** OF $f(x)$. THE INDEFINITE INTEGRAL OF $f(x)$ IS DENOTED BY $\int f(x)dx$ READ AS "THE INTEGRAL OF $f(x)$ WITH RESPECT TO x".

- ✓ THE SYMBOL $\int$ IS SAID TO BE THE INTEGRAL SIGN.
- ✓ THE FUNCTION $f(x)$ IS SAID TO BE THE INTEGRAND OF THE INTEGRAL.
- ✓ dx DENOTES THAT THE VARIABLE OF INTEGRATION IS x.
- ✓ IF A FUNCTION HAS AN INTEGRAL, THEN IT IS SAID TO BE INTEGRABLE.
- ✓ IF $F'(x) = f(x)$, THEN $\int f(x)dx = F(x) + c$
- ✓ $\int f(x)dx$ IS READ AS, "THE INTEGRAL OF $f(x)$ WITH RESPECT TO x".
- ✓ c IS SAID TO BE THE CONSTANT OF INTEGRATION.

Example 1 $\int x\,dx = \frac{x^2}{2} + c$ BECAUSE $\frac{d}{dx}\left(\frac{x^2}{2} + c\right) = \frac{2x}{2} + 0 = x$

Note:

I $\int f'(x)dx = f(x)+c$ **II** $\int \frac{d}{dx} f(x)dx = f(x)+c$

III $\frac{d}{dx}\int f(x)dx = f(x)$

Example 2 $\int \frac{d}{dx}(4x+5)\,dx = \int 4dx = 4x+c$ BECAUSE $\frac{d}{dx}(4x+c) = 4$

Example 3 YOU KNOW THAT $\frac{d}{dx}(x^6) = 6x^5 \Rightarrow \frac{1}{6}\frac{d}{dx}(x^6) = x^5$

$$\Rightarrow \int x^5 dx = \int \frac{1}{6}\frac{d}{dx}(x^6)\,dx$$

$$= \int \frac{d}{dx}\left(\frac{x^6}{6}\right)dx = \frac{x^6}{6}+c$$

AGAIN $\frac{d}{dx}\int x^5\,dx = \frac{d}{dx}\left(\frac{x^6}{6}+c\right) = x^5$

Integration of some simple functions

ACTIVITY 5.2

1 COPY AND FILL IN THE FOLLOWING TABLE

$f(x)$	4	x	x^2	x^3	x^{10}	x^n	SIN x	COS x	TAN x	COT x	e^x	4^x	LN x	LOG x
$f'(x)$														

2 BY OBSERVING THE TABLE IN PROBLEM 1 ABOVE, EVALUATE EACH OF THE FOLLOWING INTEGRALS.

A $\int x^4\,dx$ **B** $\int \text{SIN}\,x\,dx$ **C** $\int \text{COS}\,x\,dx$

D $\int \text{SEC}^2 x\,dx$ **E** $\int \text{CSC}^2 x\,dx$ **F** $\int e^x\,dx$

G $\int 4^x\,dx$ **H** $\int \frac{1}{x}\,dx$ **I** $\int \frac{1}{x\,\text{LN}\,10}\,dx$

IN THIS SECTION, YOU WILL SEE HOW TO FIND THE INTEGRALS OF CONSTANT, POWER, EXP AND LOGARITHMIC FUNCTIONS AND SIMPLE TRIGONOMETRIC FUNCTIONS.

The integration of a constant function

$\int 0\,dx = c$, *where c is a constant.*

$\int c\,dx = cx + d$, *where c is a given constant and d is the constant of integration.*

When $c = 1$, $\int dx = x + d$.

Integrating x^n, integration of a power function

Differentiating x^{n+1} *gives* $(n+1)x^n$.

So $\int (n+1)\, x^n\, dx = x^{n+1} + c$

Thus $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c;\ n \neq -1.$

Example 4 INTEGRATE EACH OF THE FOLLOWING FUNCTIONS WITH RESPECT TO x

A 4 **B** x^7 **C** x^{-5}

D $x^{\frac{1}{2}}$ **E** $x^{-\frac{3}{5}}$ **F** $x^{-\frac{4}{3}}\sqrt{x}$

Solution

A $\int 4\, dx = 4x + c$

B $\int x^7 dx = \dfrac{x^{7+1}}{7+1} + c = \dfrac{x^8}{8} + c$

C $\int x^{-5} dx = \dfrac{x^{-5+1}}{-5+1} + c = \dfrac{x^{-4}}{-4} + c = -\dfrac{1}{4x^4} + c$

D $\int x^{\frac{1}{2}}\, dx = \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \dfrac{2}{3}\sqrt{x^3} + c$

E $\int \dfrac{1}{x^{-\frac{3}{5}}} dx = \dfrac{x^{-\frac{3}{5}+1}}{\frac{-3}{5}+1} + c = \dfrac{x^{\frac{2}{5}}}{\frac{2}{5}} + c = \dfrac{5x^{\frac{2}{5}}}{2} + c$

F $\int x^{-\frac{4}{3}}\sqrt{x}\, dx = \int x^{\frac{-4}{3}+\frac{1}{2}} dx = \dfrac{x^{\frac{1}{6}}}{\frac{1}{6}} = 6\sqrt[6]{x}.$

LET k BE A CONSTANT AND $n \neq -1$, THEN $\int k\, x^n\, dx = \dfrac{k}{n+1} x^{n+1} + c$.

Integrating $(ax + b)^n$ with respect to x

Example 5 LET $y = (3x + 5)^{10}$, THEN USING THE SUBSTITUTION $u = 3x + 5$, WE HAVE $y = u^{10}$.

THEN, $\dfrac{dy}{dx} = \dfrac{dy}{du}.\dfrac{du}{dx} = 10u^9 \times 3 = 3 \times 10\, (3x+5)^9$

$\int 3 \times 10\, (3x+5)^9 dx = (3x+5)^{10} + c$

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IN GENERAL, BY APPLYING THE SAME TECHNIQUE AS EXAMPLE 5, YOU HAVE

$$\frac{d}{dx}(ax+b)^{n+1} = a(n+1)(ax+b)^n \text{ SO THAT}$$

$$\int a(n+1)(ax+b)^n dx = (ax+b)^{n+1} + c.$$

THUS, $\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$. WHERE $n \neq -1$ AND $a \neq 0$.

Note:

$$\int k(ax+b)^n\, dx = \frac{k}{a(n+1)}(ax+b)^{n+1} + c,\ n \neq -1 \text{ AND } a \neq 0$$

Example 6 INTEGRATE EACH OF THE FOLLOWING FUNCTIONS WITH RESPECT TO x.

A $5x^6$ **B** $\frac{1}{2x^4}$ **C** $(2x-11)^{11}$ **D** $(4x+3)^8$

E $5(2-3x)^{\frac{1}{2}}$ **F** $4\sqrt[3]{1-x}^{5}$ **G** $(3x+5)^3\sqrt{3x+5}$

Solution

A USING $\int kx^n dx = \frac{k}{n+1}x^{n+1} + c$, YOU GET $\int 5x^6 dx = \frac{5}{7}x^7 + c$

B $\int \frac{1}{2x^4}dx = \int \frac{1}{2}x^{-4}\, dx = \frac{1}{2}\left(\frac{x^{-3}}{-3}\right) + c = -\frac{1}{6x^3} + c$

C $\int (2x-1)^{11}dx = \frac{1}{2(11+1)}(2x-1)^{11+1} = \frac{(2x-1)^{12}}{24} + c$

D $\int (4x+3)^8 dx = \frac{1}{4\times 9}(4x+3)^9 + c = \frac{(4x+3)^9}{36} + c$

E $\int 5(2-3x)^{\frac{1}{2}}dx$. HERE $k = 5$, $a = -3$, $n = \frac{1}{2}$

HENCE, $\int 5(2-3x)^{\frac{1}{2}}\, dx = \frac{5}{(-3)\left(\frac{1}{2}+1\right)}(2-3x)^{\frac{1}{2}+1} + c$

$$= -\frac{10}{9}(2-3x)\sqrt{2-3x} + c$$

F $\int 4\sqrt[3]{(1-x)^5}dx = \frac{4}{-1\left(\frac{5}{3}+1\right)}(1-x)^{\frac{5}{3}+1} + c = -\frac{3}{2}(1-x)^2\sqrt[3]{(1-x)^2} + c$

G $\int (3x+5)^3\sqrt{3x+5}\, dx = \int (3x+5)^{\frac{7}{2}}\, dx = \frac{(3x+5)^{\frac{9}{2}}}{3\times\frac{9}{2}} + c = \frac{2(3x+5)^4\sqrt{3x+5}}{27} + c$

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Exercise 5.1

INTEGRATE EACH OF THE FOLLOWING EXPRESSIONS WITH RESPECT TO x.

1	x^3	**2**	$2x^4$	**3**	x^{-3}	**4**	$x^{\frac{2}{5}}$
5	$\frac{4}{x^{1.5}}$	**6**	$6x^2\sqrt{x}$	**7**	$\frac{1}{8\sqrt[3]{x}}$	**8**	$(3x-1)^6$
9	$\sqrt[3]{1-2x}$	**10**	$8\sqrt[4]{4-3x^3}$	**11**	$\frac{3}{\sqrt[4]{4-5x}}$	**12**	$(2x-3)^{\frac{1}{2}}$
13	$(4x- \ )^{\sqrt{2}}$						

Integration of exponential functions

YOU SHOULD REMEMBER THAT $\frac{d}{dx}e^x = e^x$

HENCE, $\int e^x dx = e^x + c$. ALSO, $\frac{d}{dx}(ke^x) = ke^x$, HENCE $\int ke^x dx = ke^x + c$

SIMILARLY $\frac{d}{dx}e^{kx} = ke^{kx} \Rightarrow \frac{1}{k}\frac{d}{dx}e^{kx} = e^{kx}$. HENCE $\int e^{kx}dx = \frac{e^{kx}}{k} + c$

FOR $a > 0$, $\frac{d}{dx}a^x = a^x \ln a \Rightarrow \frac{1}{\ln a}\frac{d}{dx}a^x = a^x$

HENCE $\int a^x \ln a \, dx = a^x + c$

THUS, $\int a^x dx = \frac{a^x}{\ln a} + c$; $a > 0$ AND $a \neq 1$

✍Note:

$$\int ka^x dx = \frac{k}{\ln a}a^x + c \quad \text{AND} \quad \int a^{kx}dx = \frac{a^{kx}}{k \ln a} + c$$

Example 7 INTEGRATE EACH OF THE FOLLOWING EXPRESSIONS WITH RESPECT TO x.

A	$3e^x$	**B**	e^{2x}	**C**	2^x	**D**	e^{-x}
E	$5\,e^{1-2x}$	**F**	3^{4+2x}	**G**	$3\,e^{4+3x}$	**H**	$\sqrt{e^x}$

Solution

A $\int 3e^x dx = 3e^x + c$

B $\int e^{2x}dx = \frac{e^{2x}}{2} + c$

C $\int 2^x dx = \frac{2^x}{\ln 2} + c$

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D $\int e^{-x}dx = \int e^{(-1)x}dx = \frac{e^{-x}}{-1} + c = -e^{x} + c$

E $\int 5e^{1-2x}dx = \int 5e \times e^{-2x}dx = 5e\left(\frac{e^{-2x}}{-2}\right) + c = \frac{-5e^{1-2x}}{2} + c$

F $\int 3^{4+2x}dx = \int 3^{4} \times 3^{2x}dx = \int 81 \times 9^{x}dx = \frac{81 \times 9^{x}}{\text{LN } 9} + c$

G $\int 3e^{4+3x}dx = \int 3e^{4} \times e^{3x}dx = 3e^{4} \times \frac{e^{3x}}{3} + c = e^{4+3x} + c$

H $\int \sqrt{e^{x}}dx = \int e^{\frac{1}{2}x}dx = \frac{e^{\frac{1}{2}x}}{\frac{1}{2}} + c = 2e^{\frac{1}{2}x} + c = 2\sqrt{e^{x}} + c$

Exercise 5.2

FIND THE INTEGRAL OF EACH OF THE FOLLOWING EXPRESSIONS WITH RESPECT TO x.

1 e^{3x} **2** e^{-5x} **3** 5^{x+1} **4** 2^{4-x} **5** e^{2-3x}

6 $4\,e^{-1-2x}$ **7** $\frac{5}{e^{+x}}$ **8** $\sqrt{3}^{\,x+5}$ **9** $\frac{4e^{4}}{e^{4x+1}}$ **10** $\sqrt{e^{2x}}$

11 4^{3x-5} **12** $\frac{2^{1-3x}}{3^{x+1}}$ **13** $2^{x+3} \times 3^{4-2x}$

Integration of $\frac{1}{x}$

IN $\int x^{n}dx$, YOU PUT A RESTRICTION $n \neq -1$. THUS, INTEGRATING $\int \frac{1}{x}dx = \int x^{-1}dx$ CANNOT BE DONE USING THE RULE OF $\int x^{n}dx = \frac{x^{n+1}}{n+1} + c$. YOU RECALL THAT FOR $x > 0$, $\frac{d}{dx}\text{LN } x = \frac{1}{x}$

$$\Rightarrow \int \frac{1}{x}dx = \text{LN } x + c$$

WHAT HAPPENS IF $x < 0$? LET $x < 0$, THEN $-x > 0$ SO THAT LN $(-x)$ IS DEFINED.

MOREOVER, $\frac{d}{dx}\text{LN}(-x) = \frac{1}{-x}\frac{d}{dx}(-x) = \frac{-1}{-x} = \frac{1}{x}$ BY THE CHAIN RULE.

$$\Rightarrow \text{FOR } x < 0, \int \frac{1}{x}dx = \text{LN}(-x) + c$$

THUS, $\int \frac{1}{x}dx = \begin{cases} \text{LN } x + c, & \text{IF } x > 0 \\ \text{LN}(-x) + c, & \text{IF } x < 0 \end{cases} \Rightarrow \int \frac{1}{x}dx = \text{LN}|x| + c$

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Note:

IF k IS A CONSTANT, THEN $\int \frac{k}{x}\,dx = k\ln|x| + c$

Example 8 EVALUATE

A $\int \frac{3}{x}\,dx$ **B** $\int \frac{1}{2x}\,dx$

Solution

A USING $\int \frac{k}{x}\,dx = k\ln|x| + c$ YOU OBTAIN $\int \frac{3}{x}\,dx = 3\ln|x| + c$

B $\int \frac{1}{2x}\,dx$, HERE $k = \frac{1}{2}$

HENCE, $\int \frac{1}{2x}\,dx = \frac{1}{2}\ln|x| + c = \ln\sqrt{|x|} + c$

NOW CONSIDER THE DERIVATIVE OF $\ln(ax+b)$ WITH RESPECT TO x, WHERE $a \neq 0$.

$$\frac{d}{dx}\ln(ax+b) = \frac{1}{ax+b} \times \frac{d}{dx}(ax+b) \quad \textit{(by the chain rule)}$$

$$= \frac{a}{ax+b} \Rightarrow \frac{1}{a}\frac{d}{dx}\ln(ax+b) = \frac{1}{ax+b}$$

$$\Rightarrow \int \frac{1}{a}\frac{d}{dx}(\ln(ax+b))\,dx = \int \frac{1}{ax+b}\,dx$$

$$\Rightarrow \int \frac{d}{dx}\left(\frac{1}{a}\ln(ax+b)\right)dx = \int \frac{1}{ax+b}\,dx$$

$$\Rightarrow \int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

Example 9 EVALUATE EACH OF THE FOLLOWING INTEGRALS.

A $\int \frac{1}{4x+1}\,dx$ **B** $\int \frac{5}{2-3x}\,dx$

Solution

A USING $\int \frac{1}{ax+b}\,dx = \frac{\ln(ax+b)}{a} + c$, YOU HAVE

$$\int \frac{1}{4x+1}\,dx = \frac{\ln(4x+1)}{4} + c$$

B $\int \frac{5}{2-3x}\,dx = \frac{5\ln|2-3x|}{-3} + c = -\frac{5}{3}\ln|2-3x| + c$

NOTE THAT, $\int \frac{1}{x}\,dx = \ln|x| + c = \ln|x| + \ln e^c = \ln|x|e^c = \ln|Ax|$; $A = e^c$

Example 10 EVALUATE THE INTEGRAL $\int \frac{1}{3x-1}\,dx$,

Solution $\int \frac{1}{3x-1}\,dx = \frac{\ln|3x-1|}{3} + c = \frac{1}{3}\ln|3x-1| + \ln e^{c} = \ln\left|\sqrt[3]{3x-1}\; e^{c}\right|$

$= \ln\left|A\sqrt[3]{3x-1}\right|, A = e^{c}$

Exercise 5.3

INTEGRATE EACH OF THE FOLLOWING EXPRESSIONS WITH RESPECT TO x.

1	$\frac{1}{3x}$	**2**	$\frac{5}{2x}$	**3**	$\frac{2}{x+1}$
4	$\frac{3}{2x-1}$	**5**	$\frac{\sqrt{2}}{1-3x}$	**6**	$\frac{4}{x-1}$
7	$\frac{-3}{5-2x}$	**8**	$\frac{1}{\left(\frac{1}{2}x+1\right)}$	**9**	$\frac{1}{x(x+1)}$

5.1.2 Properties of Indefinite Integrals

ACTIVITY 5.3

1 EVALUATE EACH OF THE FOLLOWING INTEGRALS.

A $\int\left(x^{2}+\sqrt{x}-e^{x}\right)dx$ **B** $\int x^{2}\,dx+\int\sqrt{x}\,dx-\int e^{x}dx$

C $\int(2x-1)^{2}dx$ **D** $4\int x^{2}dx-4\int x\,dx+\int dx$

E $\int\left(3x^{\frac{1}{2}}+x^{\frac{3}{2}}+\frac{1}{\sqrt{x}}-e^{-x}\right)dx$ **F** $3\int x^{\frac{1}{2}}\,dx+\int x^{\frac{3}{2}}\,dx+\int\frac{1}{\sqrt{x}}\,dx-\int e^{-x}dx$

2 YOU REMEMBER THAT DIFFERENTIATION IS A DISTRIBUTIVE PROCESS OVER ADDITION. IS INTEGRATION DISTRIBUTIVE IN THE SAME WAY? JUSTIFY YOUR ANSWER BY CONSIDERING THE INTEGRALS IN PROBLEM 1 ABOVE.

3 YOU KNOW THAT SEVERAL FUNCTIONS MAY HAVE THE SAME DERIVATIVE, FOR INSTANCE, x^2, x^2+5, x^2+2, x^2+3,... HAVE DERIVATIVE $2x$. WHAT IS THE GEOMETRICAL INTERPRETATION OF $\int 2x\,dx$?

USING ACTIVITY 5.3 AND WHAT YOU HAVE DONE SO FAR, YOU HAVE THE FOLLOWING PROPERTIES OF THE INDEFINITE INTEGRAL.

Properties of the Indefinite Integral

1. $\int f'(x)\,dx = f(x) + c$ OR $\int \frac{d}{dx} f(x)\,dx = f(x) + c.$
2. $\frac{d}{dx}\int f(x)\,dx = f(x).$
3. $\int kf(x)\,dx = k\int f(x)\,dx.$
4. $\int (f(x) + g(x))\,dx = \int f(x)dx + \int g(x)dx$
5. $\int (f(x) - g(x))\,dx = \int f(x)dx - \int g(x)dx.$

Theorem 5.1

IF TWO FUNCTIONS $F(x)$ AND $G(x)$ ARE ANTI DERIVATIVES OF THE FUNCTION $f(x)$ ON THE INTERVAL $[a, b]$, THEN $F(x) = G(x) + c$ FOR ALL $x \in [a, b]$, WHERE c IS AN ARBITRARY CONSTANT.

Proof: $(F(x) - G(x))' = F'(x) - G'(x) = f(x) - f(x) = 0$

$$\Rightarrow F(x) - G(x) = c \Rightarrow F(x) = G(x) + c$$

WE WILL EXPLAIN BRIEFLY WHAT WE MEAN BY ARBITRARY CONSTANT c.

$$\int f(x)\,dx = G(x) + c$$

IF YOU DRAW ONE OF THE INTEGRAL CURVES $y = F(x)$ BY TAKING $c = 0$, ALL THE OTHER INTEGRAL CURVES $y = F(x) + c$ ARE OBTAINED BY SHIFTING THE CURVE $y = F(x)$ IN THE y-DIRECTION. THUS YOU OBTAIN A FAMILY OF (PARALLEL) CURVES.

THE FACT THAT THEY ARE PARALLEL CURVES MEANS THAT THEY HAVE EQUAL SLOPE AT $(a, F(a) + c)$. LOOK AT FIGURE 5.1 AND FIGURE 5.2

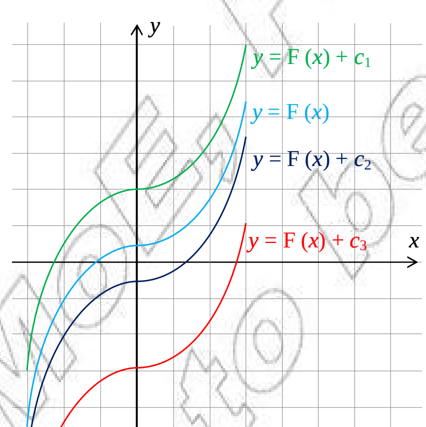

Figure 5.1

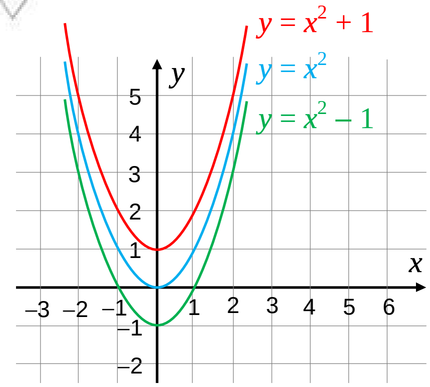

Figure 5.2

THE SLOPE OF EACH CURVE AT a IS $F'(a)$. WHERE $F'(a) = F_1'(a) = F_2'(a) = \ldots$

Example 11 LET $f(x) = 2x$. THEN $\int f(x)\,dx = x^2 + c$

THE SLOPE OF $y = x^2$ AT $x = 1$ IS $\frac{dy}{dx}\Big|_{x=1} = 2(1) = 2$

SIMILARLY, THE SLOPE OF $y = x^2 - 1$ AT $x = 1$ IS 2, AND THE SLOPE OF $y = x^2 + 1$ AT $x = 1$ IS 2. [*See* FIGURE 5.2]

Exercise 5.4

EVALUATE EACH OF THE FOLLOWING INTEGRALS.

1. $\int \frac{d}{dx}(x^3)\,dx$
2. $\frac{d}{dx}\int x^3 dx$
3. $\int \left(x^6 + x^{\frac{1}{3}} - x^{-4} + x^{-\frac{3}{2}}\right)dx$
4. $\int \left(\sqrt{x} - 3x^3 + x^{-2} + 2\right)dx$
5. $\int \frac{x^3 + x^2 + x + 1}{x^4}\,dx$
6. $\int \frac{(x+1)^2}{\sqrt{x}}\,dx$
7. $\int \frac{(z^4 + z^3 - 2z^2 + z + 1)}{z^2}\,dz$
8. $\int (x-1)(x^2 + x + 1)\,dx$
9. $\int \frac{(T^2 - 3T + 4)}{T}\,dT$
10. $\int \left(\frac{x+1}{x^2}\right)dx$
11. $\int \left(e^x - e^{-x} + \frac{1}{x}\right)dx$
12. $\int \frac{(e^x - 1)(e^x - 2)}{\sqrt{e^x}}\,dx$
13. $\int \left(2x^3 + e^{2x} - \frac{1}{2x}\right)dx$
14. $\int e^x (1 - e^x)^2\,dx$
15. $\int \left(3^{1-2x} + \frac{1}{\sqrt{2^x}} + \frac{1}{e^{2x}}\right)dx$

Integration of simple trigonometric functions

YOU KNOW THAT $\int \frac{d}{dx} f(x)\,dx = f(x) + c$

FROM ACTIVITY 5.2 YOU OBSERVED THAT $\frac{d}{dx}(\text{SIN } x) = \text{COS } x$

$$\Rightarrow \int \frac{d}{dx}(\text{SIN } x)\,dx = \int \text{COS } x\,dx$$

$$\Rightarrow \int \text{COS } x\,dx = \text{SIN } x + c$$

THEREFORE, USING THE DERIVATIVES OF SIMPLE TRIGONOMETRIC FUNCTIONS YOU OBTAIN,

$$\int \sin x \, dx = -\cos x + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

$$\int \csc^2 x \, dx = -\cot x + c$$

SIMILARLY, $\frac{d}{dx}(\sec x) = \sec x \tan x$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

THUS, $\int \sec x \tan x \, dx = \sec x + c$

$$\int \csc x \cot x \, dx = -\csc x + c$$

USING THE PROPERTIES OF INDEFINITE INTEGRALS, YOU HAVE THE FOLLOWING INTE TRIGONOMETRIC FUNCTIONS.

$$\int k \sin x \, dx = -k \cos x + c$$

$$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c \text{ ; WHERE } a \neq 0$$

Example 12 $\int \cos(5x)\,dx = \frac{1}{5}\sin(5x) + c$ BECAUSE

$$\frac{d}{ax}\left(\frac{1}{5}\sin(5x) + c\right) = \frac{1}{5}\frac{d}{dx}\sin(5x) = \frac{1}{5} \times \cos(5x) \times 5 = \cos(5x).$$

Example 13 $\int \sec^2(3x+7)\,dx = \frac{1}{3}\tan(3x+7) + c$ BECAUSE

$$\frac{d}{ax}\left(\frac{1}{3}\tan(3x+7) + c\right) = \frac{1}{3}\frac{d}{dx}\tan(3x+7) = \frac{1}{3}\sec^2(3x+7) \times 3 = \sec^2(3x+7).$$

Exercise 5.5

INTEGRATE EACH OF THE FOLLOWING EXPRESSIONS WITH RESPECT TO x.

1	$3\sin(x)$	**2**	$\cos(2x)$	**3**	$\sin(4x-1)$
4	$3\cos(4x+\frac{\pi}{3})$	**5**	$\sin(3x) + \cos(4x)$	**6**	$\sec^2(2x+1)$
7	$\csc(2x)\cot(2x)$	**8**	$\sec\left(2x-\frac{\pi}{4}\right)\tan\left(2x-\frac{\pi}{4}\right)$		

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5.2 TECHNIQUES OF INTEGRATION

IN DIFFERENTIAL CALCULUS YOU HAVE SEEN DIFFERENT RULES SUCH AS: THE ADDITION, PRODUCT, QUOTIENT AND CHAIN RULES. ALSO, IN THE REVERSE PROCESS, INTEGRATION, DIFFERENT METHODS. THE MOST COMMONLY USED METHODS ARE: SUBSTITUTION, PARTIAL AND INTEGRATION BY PARTS.

5.2.1 Integration by Substitution

Integration by substitution IS A COUNTER PART TO THE **chain rule of differentiation**. IT IS A METHOD OF FINDING INTEGRALS BY CHANGING VARIABLES. THE INTEGRAL EXPRESSED VARIABLE MAY BE SIMPLER TO EVALUATE OR CHANGED FROM THE UNFAMILIAR INTEGRA BETTER UNDERSTOOD FORM. THIS METHOD IS BASED ON A CHANGE OF VARIABLE EQUATIO CHAIN RULE. THE CHANGE OF THE VARIABLE IS HELPFUL TO MAKE UNFAMILIAR INTEGRAL INTEGRAL FORM YOU CAN RECOGNIZE.

CONSIDER $\int 2x\left(x^2+1\right)^5 dx$

LET $u = x^2 + 1$, THEN $\frac{du}{dx} = \frac{d}{dx}\left(x^2+1\right) = 2x \Rightarrow du = 2x\,dx$

$$\Rightarrow \int 2x\left(x^2+1\right)^5 dx = \int u^5 du = \frac{u^6}{6} + c$$

BUT, $u = x^2 + 1$.

THUS, $\int 2x\left(x^2+1\right)^5 dx = \frac{\left(x^2+1\right)^6}{6} + c$

IN THIS INTEGRATION, YOU CHANGE THE VARIABLE FROM x TO u.

YOU REMEMBER THAT IF u IS A FUNCTION OF x, THEN FOR A FUNCTION $f(u)$,

$$\frac{d}{dx} f(u) = \frac{du}{dx} f'(u) \Rightarrow \int \frac{d}{dx} f(u)\,du = f(u) + c$$

$$\Rightarrow \int \frac{du}{dx} f'(u)\,du = f(u) + c \Rightarrow \int f'(u)\frac{du}{dx}dx = \int f'(u)du$$

Example 1 FIND $\int x\sqrt{x^2+5}\,dx$

Solution LET $u = x^2 + 5$, THEN $\frac{du}{dx} = \frac{d}{dx}\left(x^2+5\right) = 2x \Rightarrow \frac{1}{2}\,du = x\,dx$

HENCE, $\int x\sqrt{x^2+5}\,dx = \int \sqrt{x^2+5}\,x\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{1}{3}u\sqrt{u} + c$

$$\Rightarrow \int x\sqrt{x^2+5}\,dx = \frac{1}{3}\left(x^2+5\right)\sqrt{x^2+5} + c$$

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Example 2 FOR EACH OF THE FOLLOWING EXPRESSIONS, SUGGEST SUITABLE VARIABLE OF SUBSTITUTION AND INTEGRATE WITH RESPECT TO x.

A $x^2(5x^3 - 2)^9$ **B** $\cos x\, e^{\sin x}$ **C** xe^{x^2}

D $\dfrac{x}{x^2+7}$ **E** $\cos^3 x \sin x$ **F** $\sqrt{x}\sqrt{1+x\sqrt{x}}$

Solution REWRITE THE INTEGRAL USING u AS THE VARIABLE OF SUBSTITUTION.

A $\int x^2\left(5x^3-2\right)^9 dx.$

HERE, THE FACTOR OF THE INTEGRAND, x^2, IS THE DERIVATIVE OF $\frac{1}{15}\left(5x^3-2\right)$

THUS, $u = 5x^3 - 2 \Rightarrow \dfrac{du}{dx} = \dfrac{d}{dx}\left(5x^3-2\right) = 15x^2$

$$\Rightarrow \frac{1}{15}du = x^2\ dx \Rightarrow \int x^2\left(5x^3-2\right)^9 dx = \frac{1}{15}\int u^9 du = \frac{1}{15}\left(\frac{u^{10}}{10}\right)+c$$

$$\Rightarrow \int x^2\left(5x^3-2\right)^9 dx = \frac{1}{150}\left(5x^3-2\right)^{10}+c$$

B $\int \cos x\, e^{\sin x} dx$

YOU KNOW THAT $\dfrac{d}{dx}(\sin x) = \cos x$

HENCE, $u = \sin x \Rightarrow du = \cos x\, dx$

$$\Rightarrow \int \cos x\, e^{\sin x} dx = \int e^u du = e^u + c \Rightarrow \int \cos x\, e^{\sin x} dx = e^{\sin x} + c$$

C $\int x\, e^{x^2}\, dx$

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow \frac{1}{2}du = x\ dx$$

$$\Rightarrow \int x e^{x^2} dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + c \Rightarrow \int x e^{x^2} dx = \frac{1}{2}e^{x^2}+c$$

ALSO, OBSERVE THAT, $\dfrac{d}{dx}\left(e^{x^2}\right) = 2xe^{x^2}$

HENCE, $u = e^{x^2} \Rightarrow \dfrac{du}{dx} = 2xe^{x^2}$

$$\Rightarrow \frac{1}{2}du = xe^{x^2}dx \Rightarrow \int x e^{x^2} dx = \frac{1}{2}\int du = \frac{1}{2}u + c$$

$$\Rightarrow \int x e^{x^2} dx = \frac{1}{2}e^{x^2}+c$$

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D $\int \frac{x}{x^2+7}\,dx$; $u = x^2+7$

$$\Rightarrow \frac{du}{dx} = 2x \Rightarrow \frac{1}{2}du = x\,dx \Rightarrow \int \frac{x}{x^2+7}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + c$$

$$\Rightarrow \int \frac{x}{x^2+7}\,dx = \frac{1}{2}\ln\left|x^2+7\right| + c = \ln\sqrt{x^2+7} + c$$

E $\int \cos^3 x \sin x\,dx$

$$u = \cos x \Rightarrow \frac{du}{dx} = -\sin x \Rightarrow -du = \sin x\,dx$$

$$\Rightarrow \int \cos^3 x \sin x\,dx = -\int u^3\,du = \frac{-u^4}{4} + c \Rightarrow \int \cos^3 x \sin x\,dx = \frac{-\cos^4 x}{4} + c$$

F $\int \sqrt{x}\sqrt{1+x\sqrt{x}}\,dx$

$$u = 1 + x\sqrt{x} \Rightarrow \frac{du}{dx} = \frac{d}{dx}\left(1+x^{\frac{3}{2}}\right) = \frac{3}{2}x^{\frac{1}{2}} \Rightarrow \frac{2}{3}du = \sqrt{x}\,dx$$

$$\int \sqrt{x}\sqrt{1+x\sqrt{x}}\,dx = \frac{2}{3}\int \sqrt{u}\,du = \frac{2}{3}\left(\frac{2}{3}\right)u^{\frac{3}{2}} + c = \frac{4}{9}u^{\frac{3}{2}} + c = \frac{4}{9}\left(1+x\sqrt{x}\right)^{\frac{3}{2}} + c$$

Example 3 Find $\int (3x-2)\sqrt{x+6}\,dx$.

Solution Here, $3x - 2$ is not a constant times the derivative of $x + 6$ or vice versa. But you can still use substitution as follows.

$u = x + 6 \Rightarrow x = u - 6 \Rightarrow 3x - 2 = 3(u - 6) - 2 = 3u - 20$; $u = x + 6 \Rightarrow du = dx$

Thus, $\int (3x-2)\sqrt{x+6}\,dx = \int (3u-20)\sqrt{u}\,du = \int 3u\sqrt{u} - 20\sqrt{u}\,du$

$$= 3\int u^{\frac{3}{2}}\,du - 20\int u^{\frac{1}{2}}\,du = 3\frac{u^{\frac{5}{2}}}{\frac{5}{2}} + c_1 - 20\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c_2 = \frac{6}{5}u^2\sqrt{u} - \frac{40}{3}u\sqrt{u} + c$$

$$\Rightarrow \int (3x-2)\sqrt{x+6}\,dx = \frac{6}{5}(x+6)^2\sqrt{x+6} - \frac{40}{3}(x+6)\sqrt{x+6} + c.$$

Example 4 Evaluate $\int \frac{3}{2x+1}\,dx$.

Solution $u = 2x + 1 \Rightarrow \frac{1}{2}du = dx$

$$\Rightarrow \int \frac{3}{2x+1}\,dx = \frac{3}{2}\int \frac{1}{u}\,du = \frac{3}{2}\ln|u| + c \Rightarrow \int \frac{3}{2x+1}\,dx = \frac{3}{2}\ln|2x+1| + c$$

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Example 5 EVALUATE THE FOLLOWING INTEGRALS

A $\int f(x) f'(x)\,dx$ **B** $\int \frac{f'(x)}{f(x)}\,dx,\ f(x) \neq 0$

Solution $u = f(x) \Rightarrow \frac{du}{dx} = \frac{d}{dx} f(x) = f'(x) \Rightarrow du = f'(x)\,dx$

A $\int f(x) f'(x)\,dx = \int u\,du = \frac{u^2}{2} + c \Rightarrow \int f(x) f'(x)\,dx = \frac{(f(x))^2}{2} + c$

B $\int \frac{f'(x)}{f(x)}\,dx = \int \frac{1}{u}\,du = \ln|u| + c \Rightarrow \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

Example 6 USING $\int \frac{f'(x)}{f(x)}\,dx$, SHOW THAT $\int \tan x\,dx = -\ln|\cos x| + c$

Solution $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{(\cos x)'}{\cos x}\,dx = -\ln|\cos x| + c$

Example 7 USING A SUITABLE IDENTITY, FIND $\int \sin^2 x\,dx$

Solution BY WRITING $\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$, YOU HAVE,

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\Rightarrow \int \sin^2 x\,dx = \int \frac{1 - \cos(2x)}{2}\,dx = \int \frac{1}{2}\,dx - \frac{1}{2}\int \cos(2x)\,dx$$

BUT $\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + c$ *Explain!*

$$\Rightarrow \int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + c$$

Example 8 FIND $\int 2^{4x-1}\,dx$ USING THE METHOD OF SUBSTITUTION.

Solution $u = 4x - 1 \Rightarrow \frac{du}{dx} = 4 \Rightarrow \frac{1}{4}du = dx$

$$\Rightarrow \int 2^{4x-1}\,dx = \frac{1}{4}\int 2^u\,du = \frac{1}{4}\left(\frac{2^u}{\ln 2}\right) = \frac{2^u}{\ln 16} + c \Rightarrow \int 2^{4x-1}\,dx = \frac{2^{4x-1}}{\ln 16} + c.$$

Can you do this without using substitution?

LOOK AT THE FOLLOWING.

$$\int 2^{4x-1}\,dx = \int \frac{16^x}{2}\,dx = \frac{1}{2}\int 16^x\,dx = \frac{1}{2}\left(\frac{16^x}{\ln 16}\right) + c = \frac{2^{4x-1}}{\ln 16} + c$$

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Example 9 FIND $\int x^2 \cos(x^3+1)\,dx$

Solution $u = x^3+1 \Rightarrow \frac{1}{3}du = x^2dx \Rightarrow \int x^2\cos(x^3+1)\,dx = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + c$

$$\int x^2\cos(x^3+1)\,dx = \frac{1}{3}\sin(x^3+1)+c$$

Example 10 EVALUATE $\int \frac{x}{\sqrt{x^2+a^2}}\,dx$

Solution LET $u = x^2+a^2 \Rightarrow \frac{du}{dx} = \frac{d}{dx}\left(x^2+a^2\right) = 2x \Rightarrow \frac{1}{2}du = x\,dx$

THUS, $\int \frac{x}{\sqrt{x^2+a^2}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u}+c \Rightarrow \int \frac{x}{\sqrt{x^2+a^2}}\,dx = \sqrt{x^2+a^2}+c.$

Example 11 EVALUATE $\int \frac{1}{x\ln x}\,dx$

Solution IN THE PRODUCT $\left(\frac{1}{x}\right)\left(\frac{1}{\ln x}\right)$, THE FACTOR $\frac{1}{x}$ IS THE DERIVATIVE OF $\ln x$.

THEREFORE, LET $u = \ln x$ SO THAT $\int \frac{du}{u} = \ln|u|+c = \ln|\ln x|+c$

Exercise 5.6

1 INTEGRATE EACH OF THE FOLLOWING EXPRESSIONS WITH RESPECT TO x.

A $2x\left(x^2+1\right)^3$ **B** $x\sqrt{x^2+4}$ **C** $x^2\sqrt{x^3+1}$

D $(2x+1)\sqrt{x^2+x+9}$ **E** $\sin x\cos x$ **F** $(2x+3)e^{(x^2+3x+4)}$

G $\sin x\, e^{\cos x}$ **H** $(x+2)\sqrt{x-3}$

2 FIND EACH OF THE FOLLOWING INTEGRALS USING THE GIVEN SUBSTITUTIONS.

A $\int\sqrt{3x-2}\,dx;\ u = 3x-2$ **B** $\int x\sqrt{1-5x^2}\,dx;\ u = 1-5x^2$

C $\int \sin(2x)\,dx;\ u = 2x$ **D** $\int(1-4x)\,dx;\ u = 1+x$

E $\int x\left(x^2-3\right)^5dx;\ u = x^2-3$ **F** $\int x^2\left(2+3x^3\right)dx;\ u = 3x^3+2$

G $\int e^x\sqrt{1+e^x}\,dx;\ u = 1+e^x$ **H** $\int \sin x\cos^{10}x\,dx;\ u = \cos x$

I $\int\sqrt{4x-3}\,dx;\ u = x-3$ **J** $\int \frac{1}{(1-x)^{\frac{1}{3}}}\,dx;\ u = 1-x$

K $\int 3^{\frac{1}{x}}\, x^{-2}dx;\ u = \frac{1}{x}$ **L** $\int 3^{0.6x+\pi}\, dx;\ u = 0.6x + \pi$

M $\int \cos(-3x)\, dx;\ u = -3x$ **N** $\int x\sin(x^2 + 7)\, dx;\ u = x^2 + 7$

O $\int \frac{4x-5}{2x^2-5x+4}\, dx;$ $u = 2x^2 - 5x + 4$ **P** $\int \frac{x+1}{\sqrt{x+3}}\, dx;\ u = x+3$ OR $u = x + 1$

Q $\int (3+2x)^{12} dx;\ u = 3+2x$ **R** $\int \tan x \sec^2 x\, dx;\ u = \tan x$

S $\int \sin(2x + \ )\, dx;\ u = 2x +$ **T** $\int 5^{x\sqrt{x}}\sqrt{x}\, dx;\ u = x\sqrt{x}$

U $\int \frac{x}{\sqrt{x^2+5}}\, dx;\ u = x^2+5$ **V** $\int (2x-3)\sqrt{x+3}\, dx;\ u = x+3$

3 EVALUATE EACH OF THE FOLLOWING INTEGRALS.

A $\int x^3\left(x^4+5\right)\, dx$ **B** $\int \left(1-\frac{1}{x^2}\right)\left(x+\frac{1}{x}\right)dx$

C $\int \left(2^{x^2}\right)x\, dx$ **D** $\int \cot x\, dx$

E $\int \sin x\sqrt{1-\cos x}\, dx$ **F** $\int e^x\sqrt{4+e^x}\, dx$

G $\int (ax+b)^n\, dx$ **H** $\int \cos(4x+3)\, dx$

I $\int 3^x\left(1-3^{(x+1)}\right)^9 dx$ **J** $\int \frac{4}{4x-2}\, dx$

K $\int \frac{1}{ax+b}\, dx$ **L** $\int \frac{1}{(ax+b)^n}\, dx$

M $\int \frac{x}{\sqrt{x^2+1}}\, dx$ **N** $\int x^2\left(x^3-8\right)\, dx$

O $\int \frac{e^{\sqrt{t}}}{\sqrt{t}}\, dt$ **P** $\int \frac{2^{\sqrt{y}}}{\sqrt{y}}\, dy$

Q $\int x\sqrt{3+5x}\, dx$ **R** $\int \frac{\sin t}{\sqrt{3+\cos t}}\, dt$

S $\int \frac{\sin(2t)}{1-\cos(2t)}\, dt$ **T** $\int xe^{x^2+7}dx$

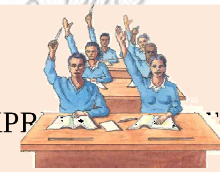
U $\int \frac{4x-1}{1-2x+4x^2}\, dx$ **V** $\int (3x+1)\left(3x^2+2x+5\right)^6 dx$

W $\int (x-1)\sqrt[3]{\left(x^2-2x+3\right)^2}\, dx$ **X** $\int \cos x \sin^{10} x\, dx$

5.2.2 Integration by partial fractions

DECOMPOSITION OF A RATIONAL EXPRESSION INTO PARTIAL FRACTIONS WAS DISCUSSED IN
IN THIS SECTION, TO FIND THE INTEGRALS OF SOME RATIONAL EXPRESSIONS, YOU USE
FRACTIONS ALONG WITH THE METHOD OF SUBSTITUTION.

ACTIVITY 5.4



1 DECOMPOSE EACH OF THE FOLLOWING RATIONAL EXPR O PARTIAL FRACTIONS

A $\dfrac{1}{x(x+1)}$ **B** $\dfrac{x}{x^3-3x+2}$ **C** $\dfrac{2x-3}{(x-1)^2}$

D $\dfrac{x^3}{x^2-4x+3}$ **E** $\dfrac{x+2}{x^2(x-3)}$ **F** $\dfrac{x^2+2x+3}{(x+1)(x^2-4)}$

G $\dfrac{x-1}{(x+1)^2(x+2)}$

2 CONSIDER THE INTEGRAL OF THE RATIONAL EXPRESSION $\dfrac{x+3}{x+1}$, REWRITE THIS EXPRESSION AS $1+\dfrac{2}{x+1}$ BY USING LONG DIVISION.

$$\Rightarrow \int \frac{x+3}{x+1}dx=\int\left(1+\frac{2}{x+1}\right)dx = x+2\int\frac{1}{x+1}dx = x+2\ln|x+1|+c$$

USING THIS TECHNIQUE OF INTEGRATION, FIND EACH OF THE FOLLOWING INTEGRALS.

A $\int\dfrac{x+2}{x+3}\,dx$ **B** $\int\dfrac{x+2}{4x-3}\,dx$ **C** $\int\dfrac{x}{4x+5}\,dx$

D $\int\dfrac{4x-5}{5x-4}\,dx$ **E** $\int\dfrac{1}{(2x-1)^4}\,dx$ **F** $\int\left(\dfrac{x+1}{x-3}\right)^3dx$

3 YOU KNOW THAT $\int\left(\dfrac{1}{x+2}+\dfrac{3}{x-1}\right)dx=\int\dfrac{1}{x+2}dx+\int\dfrac{3}{x-1}dx=\ln|x+2|+3\ln|x-1|+c$

CAN YOU EVALUATE THIS INTEGRAL BY SUMMING UP THE EXPRESSIONS?

$$\text{I.E., } \int\left(\frac{1}{x+2}+\frac{3}{x-1}\right)dx=\int\frac{x-1+3(x+2)}{(x+2)(x-1)}dx=\int\frac{4x+5}{(x+2)(x-1)}dx$$

FROM ACTIVITY 5.4, YOU HAVE SEEN THAT DECOMPOSITION INTO PARTIAL FRACTIONS TOGETHE
SUBSTITUTION ENABLES YOU TO EVALUATE THE INTEGRALS OF SOME RATIONAL EXPRESSIO

Example 12 FIND $\int \frac{x+5}{x^2+4x+3}\,dx$

Solution USING PARTIAL FRACTIONS, YOU OBTAIN,

$$\frac{x+5}{x^2+4x+3} = \frac{A}{x+1} + \frac{B}{x+3} \Rightarrow \int \frac{x+5}{x^2+4x+3}dx = \int\left(\frac{A}{x+1} + \frac{B}{x+3}\right)dx$$

$$= A \ln|x+1| + B \ln|x+3| + c = 2 \ln|x+1| - \ln|x+3| + c$$

Example 13 FIND $\int \frac{x^3+2x^2-x-7}{x^2+x-2}\,dx.$

Solution THE RATIONAL EXPRESSION IS AN IMPROPER FRACTION, THEREFORE, BEFORE FACTORIZING THE DENOMINATOR WE USE LONG DIVISION, TO OBTAIN

$$\int \frac{x^3+2x^2-x-7}{x^2+x-2}dx = \int\left(x+1-\frac{5}{x^2+x-2}\right)dx$$

$$= \frac{x^2}{2}+x-5\int\left(\frac{A}{x+2}+\frac{B}{x-1}\right)dx = \frac{x^2}{2}+x-5\left(A\ln|x+2|+B\ln|x-1|\right)+c$$

$$= \frac{x^2}{2}+x+\frac{5}{3}\ln|x+2|-\frac{5}{3}\ln|x-1|+c = \frac{x^2}{2}+x+\frac{5}{3}\left[\ln|x+2|-\ln|x-1|\right]+c$$

$$= \frac{x^2}{2}+x+\frac{5}{3}\ln\left|\frac{x+2}{x-1}\right|+c$$

Example 14 EVALUATE $\int \frac{dx}{x^2-9}$

Solution USING PARTIAL FRACTIONS YOU HAVE

$$\int \frac{dx}{x^2-9} = \int \frac{A}{x-3}\,dx + \int \frac{B}{x+3}\,dx = A\ln|x-3| + B\ln|x+3| + c$$

FROM PARTIAL FRACTIONS WE CALCULATE THE VALUES $A = \frac{1}{6}$ AND $B = -\frac{1}{6}$

$$\Rightarrow \int \frac{dx}{x^2-9} = \frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3| + c = \ln\sqrt[6]{\left|\frac{x-3}{x+3}\right|} + c$$

Exercise 5.7

USE THE METHOD OF SUBSTITUTION ALONG WITH PARTIAL FRACTIONS TO EVALUATE E FOLLOWING INTEGRALS.

1 $\int \frac{x}{x+5}\,dx$ **2** $\int \frac{4x+1}{x^2-3x+2}\,dx$ **3** $\int \frac{x^2-x-2}{x^2+x-2}\,dx$

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4 $\int \frac{x^2+4}{x^2-1}\,dx$ **5** $\int \frac{3x+5}{x+2}\,dx$ **6** $\int \frac{x}{x^2-2x-8}\,dx$

7 $\int \frac{x}{\left(x^2-3x-8\right)^2}\,dx$ **8** $\int \frac{x^3}{(x+1)^2(x+2)}\,dx$ **9** $\int \frac{1}{(x+2)^2}\,dx$

10 $\int \frac{x^2+2x-3}{x^2\left(x^2-5x+6\right)}\,dx$

5.2.3 Integration by parts

THE PRODUCT RULE FOR DIFFERENTIATION IS

$$\frac{d}{dx}\left(f(x).g(x)\right)=g(x)\frac{d}{dx}f(x)+f(x)\frac{d}{dx}g(x)$$

THIS FORM CANNOT BE EXPRESSED AS $\frac{du}{dx}f(x)$.

HENCE, IT CANNOT BE INTEGRATED BY THE METHOD OF SUBSTITUTION.

INTEGRATION BY PARTS IS A METHOD WHICH IS A COUNTER PART OF THE PRODUCT RULE OF DIFFERENTIATION.

INTEGRATING BOTH SIDES OF THE ABOVE EXPRESSIONS GIVES,

$$\int \frac{d}{dx}\left(f(x).g(x)\right)dx=\int g(x)\frac{d}{dx}f(x)\,dx+\int f(x)\frac{d}{dx}g(x)\,dx$$

$$\Rightarrow f(x).g(x)=\int g(x)\frac{d}{dx}f(x)\,dx+\int f(x)\frac{d}{dx}g(x)\,dx$$

$$\Rightarrow \int f(x)\frac{d}{dx}g(x)\,dx=f(x).\,g(x)-\int g(x)\frac{d}{dx}f(x)\,dx.$$

ACTIVITY 5.5

1 DIFFERENTIATE EACH OF THE FOLLOWING EXPRESSIONS WITH RESPECT TO x.

A $x \ln x - x + 4$ **B** $x e^x - e^x - 7$

C $x \cos x - \cos x + 5$ **D** $e^x (\sin x + \cos x)$

E $x^2 \ln x - x^2$

2 USING THE RESULT OF PROBLEM 1 ABOVE, EVALUATE EACH OF THE FOLLOWING INTEGRALS.

A $\int \ln x\, dx$ **B** $\int x e^x dx$ **C** $\int x \sin x\, dx$

D $\int e^x \sin x\, dx$ **E** $\int x \ln x\, dx$

3 SUPPOSE YOU WANT TO EVALUATE $\int x^2 \sin x\, dx$, WHICH METHOD ARE YOU GOING TO APPLY?

✍Note:

LET u AND v BE FUNCTIONS OF x I.E. $u = u(x)$ AND $v = v(x)$.

THEN, $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx} \Rightarrow u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$

$$\Rightarrow \int u\frac{dv}{dx}\,dx = \int \frac{d}{dx}(uv)\,dx - \int v\frac{du}{dx}dx \Rightarrow \int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

IN SHORT, $\int u\,dv = uv - \int v\,du$

IN THIS METHOD, YOU SHOULD BE ABLE TO CHOOSE "PARTS" u AND dv.

Examples 15 EVALUATE $\int xe^x\,dx$

Solution HERE $xe^x = u\,dv$.

NOW, DECIDE WHICH PART SHOULD BE u AND WHICH PART SHOULD BE dv.

SUPPOSE $u = x$ AND $dv = e^x$, THEN

$\frac{du}{dx} = 1$, AND $\int dv = \int e^x dx \Rightarrow v = e^x$

$$\Rightarrow \int xe^x\,dx = uv - \int v\frac{du}{dx}\,dx = xe^x - \int e^x\,dx = xe^x - e^x + c$$

IF $u = e^x$ AND $dv = x$.

THEN, $\frac{du}{dx} = e^x$ AND $v = x^2 = \int xe^x dx = uv - \int v\frac{du}{dx}\,dx = e^x.\,x^2 - \int x^2e^x dx$

THIS IS MORE COMPLEX THAN THE ORIGINAL INTEGRAL. HENCE, IT IS SOMETIMES HELP CONSIDER u TO BE THE POLYNOMIAL FACTOR.

IN THE EXPRESSION xe^x, x IS THE POLYNOMIAL FACTOR.

Example 16 EVALUATE $\int \ln x\,dx$

Solution IN $\ln x$, WHAT IS THE POLYNOMIAL FACTOR?

LET $u = \ln x$ AND $dv = dx$. THEN, $du = \frac{1}{x}$ AND $v = x$.

THUS, $\int \ln x\,dx = x\ln x - \int x\left(\frac{1}{x}\right)dx = x\ln x - \int dx = x\ln x - x + c$

Example 17 EVALUATE $\int \log_2 x\,dx$

Solution NOTE THAT $\log_2 x = \frac{\ln x}{\ln 2}$

HENCE $\int \log_2 x\,dx = \int \frac{\ln x}{\ln 2}dx = \frac{1}{\ln 2}\int \ln x\,dx = \frac{1}{\ln 2}(x\ln x - x) + c$

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> **✍Note:**
>
> If $a > 0$ and $a \neq 1$,
>
> $$\int \log_a x\,dx = \int \frac{\ln x}{\ln a}\,dx = \frac{1}{\ln a}\int \ln x\,dx$$
>
> $$= \frac{1}{\ln a}\left(x\ln x - x\right) + c$$

Example 18 Evaluate $\int \log(3x+1)\,dx$

Solution Let $u = 3x + 1$, then

$$\frac{du}{dx} = 3 \Rightarrow \frac{1}{3}du = dx$$

$$\Rightarrow \int \log(3x+1)\,dx = \int \frac{\ln(3x+1)}{\ln 10}\,dx = \frac{1}{3\ln 10}\int \ln u\,du$$

$$= \frac{1}{3\ln 10}\left(u\ln u - u\right) + c$$

$$= \frac{1}{3\ln 10}\left((3x+1)\ln(3x+1) - (3x+1)\right) + c$$

$$= \frac{1}{3\ln 10}\left((3x+1)\ln(3x+1) - 3x - 1\right) + c$$

Example 19 Evaluate $\int x\sin x\,dx$

Solution $u = x \Rightarrow du = dx$

$$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x \Rightarrow \int x\sin x\,dx = -x\cos x - \int -\cos x\,dx$$

$$= -x\cos x + \sin x + c$$

Example 20 Evaluate $\int x\ln x\,dx$

Solution $u = \ln x \Rightarrow du = \frac{1}{x}$ and $dv = x \Rightarrow v = \frac{x^2}{2}$

$$\Rightarrow \int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} + c$$

$$= \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + c$$

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CAN YOU ASSUME $u = x$ AND $dv = \ln x$?

IF YOU SET $u = x$, THEN $du = dx$ AND $dv = \ln x\, dx$

$$\Rightarrow v = x \ln x - x$$

THEN $\int x \ln x\, dx = x(x \ln x - x) - \int (x \ln x - x)\, dx$

$$= x^2 \ln x - x^2 - \int x \ln x\, dx + \int x\, dx$$

$$\Rightarrow 2\int x \ln x\, dx = x^2 \ln x - x^2 + \frac{x^2}{2} + c$$

$$\Rightarrow \int x \ln x\, dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c$$

ALTHOUGH THIS GIVES YOU THE CORRECT ANSWER, IT IS SAFER TO SET $u = \ln x$.

Example 21 EVALUATE $\int x^r \ln x\, dx$ WHERE r IS A REAL NUMBER DIFFERENT FROM –1.

Solution WHAT HAPPENS IF $r = -1$? ARE YOU GOING TO USE BY PARTS?

IF $r = -1$, THEN $\int x^r \ln x\, dx = \int \frac{\ln x}{x}\, dx$

BY THE METHOD OF SUBSTITUTION YOU HAVE,

$u = \ln x \Rightarrow du = \frac{1}{x}\, dx,$

$$\int \frac{\ln x}{x}\, dx = \int u\, du = \frac{u^2}{2} + c \Rightarrow \int \frac{\ln x}{x}\, dx = \frac{\ln^2 x}{x} + c$$

IF $r \neq -1$, THEN $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$$dv = x^r\, dx \Rightarrow v = \frac{x^{r+1}}{r+1}.$$

THEN $\int x^r \ln x\, dx = uv - \int v\, du$

$$= (\ln x)\frac{x^{r+1}}{r+1} - \int \frac{x^{r+1}}{r+1}\left(\frac{1}{x}\right) dx$$

$$= \frac{x^{r+1}}{r+1}\ln x - \frac{x^{r+1}}{(r+1)^2} + c$$

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Example 22 EVALUATE $\int x^2 \log_3 x\,dx$

Solution $\int x^2 \log_3 x\,dx = \dfrac{1}{\ln 3}\int x^2 \ln x\,dx$

$$= \frac{1}{\ln 3}\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right] + c \quad \textit{Why?}$$

Example 23 FIND $\int e^x \sin x\,dx$

Solution CHOOSE $u = e^x$ AND $dv = \sin x$

THEN $du = e^x\,dx$ AND $v = -\cos x$.

$$\Rightarrow \int e^x \sin x\,dx = -e^x \cos x - \int -\cos x\, e^x\,dx = -e^x \cos x + \int \cos x\, e^x\,dx$$

$\int e^x \cos x\,dx$ HAS THE SAME FORM AS $\int e^x \sin x\,dx$

HENCE YOU APPLY INTEGRATION BY PARTS FOR A SECOND TIME.

$u = e^x \Rightarrow du = e^x\,dx$ AND $dv = \cos x \Rightarrow v = \sin x$

$$\Rightarrow \int \cos x\, e^x\,dx = e^x \sin x - \int \sin x\, e^x\,dx$$

BUT $\int e^x \sin x\,dx = -e^x \cos x + \int \cos x\, e^x\,dx = -e^x \cos x + e^x \sin x - \int \sin x\, e^x\,dx$

BY COLLECTING LIKE TERMS, YOU OBTAIN

$$2\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x + c$$

$$\Rightarrow \int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + c$$

In the integral $\int f(x)g(x)\,dx$, IF $f(x)$ IS A TRANSCENDENTAL FUNCTION (EXPONENTIAL, TRIGONOMETRIC OR LOGARITHMIC FUNCTION) AND $g(x)$ IS A POLYNOMIAL FUNCTION, USE THE SUBSTITUTION $u = g(x)$ AND $dv = f(x)\,dx$ for integration by parts.

Exercise 5.8

INTEGRATE EACH OF THE FOLLOWING EXPRESSIONS WITH RESPECT TO x USING THE METHOD OF INTEGRATION BY PARTS.

1	xe^{1-x}	**2**	$x\cos x$	**3**	xe^{3x+1}
4	x^2e^x	**5**	$4x\sin x$	**6**	$e^x\cos(2x)$
7	$e^{3x}\sin x$	**8**	$e^{-x}\sin(2x)$	**9**	$\ln(4x)$
10	$x^3\ln x$	**11**	$e^x(x+2)$	**12**	$x^2\sin x$
13	$x^2\ln(2x)$	**14**	$x\ln(nx)$; $n > 0$	**15**	$x\sin(nx)$; $n > 0$

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5.3 DEFINITE INTEGRALS, AREA AND THE FUNDAMENTAL THEOREM OF CALCULUS

OPENING PROBLEM

THE AREA UNDER THE CURVE OF $y = 5 - 2x^2 + 4x^3 - x^5$ FROM $x = -\frac{1}{2}$ TO $x = 1\frac{1}{2}$ IS DIVIDED INTO n STRIPS. EACH STRIP IS APPROXIMATED BY A RECTANGLE OF WIDTH $\frac{1}{n}$ AS SHOWN IN THE FOLLOWING FIGURE.

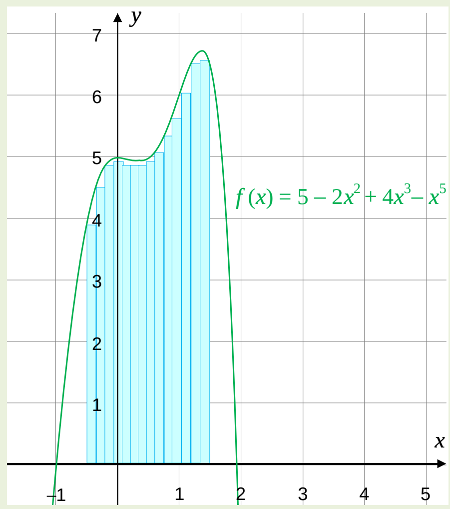


Figure 5.3

WHAT IS THE LIMIT OF THE SUM OF THE AREAS OF ALL THE RECTANGLES AS n APPROACHES INFINITY?

5.3.1 The Area of a Region under a Curve

FROM GEOMETRY, YOU KNOW HOW TO DETERMINE THE AREAS OF CERTAIN PLANE FIGURES TRIANGLES, RECTANGLES, PARALLELOGRAMS, TRAPEZIUMS, DIFFERENT REGULAR POLYGO COMBINATIONS OF PARTS OF CIRCLES AND POLYGONS.

IN THIS TOPIC, YOU SHALL DETERMINE THE AREA OF A REGION UNDER THE CURVE OF A NO FUNCTION $y = f(x)$ THAT IS CONTINUOUS ON A CLOSED INTERVAL $[a, b]$. YOU DIVIDE THE REGION INTO n STRIPES APPROXIMATED BY n RECTANGLES OF UNIFORM WIDTH Δx,

WHERE $\Delta x = \frac{b-a}{n}$ FORMED BY VERTICAL LINES THROUGH $a = x_0, x_1, x_2, \ldots, x_n = b$; WHERE

$a = x_0 < x_1 < x_2 < \ldots < x_n = b$, $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \ldots = (x_n - x_{n-1}) = \Delta x$

LOOK AT FIGURE 5.4

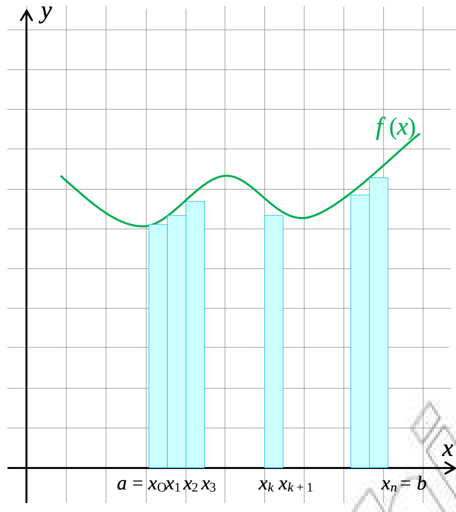
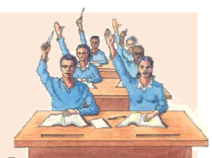


Figure 5.4

AS THE VALUE OF n GETS LARGER AND LARGER THE RECTANGLES GET THINNER AND THINNER. RECTANGLES RISE UP TO FILL IN THE REGION.

THUS, THE AREA OF THE REGION WILL BE THE LIMITING VALUE OF THE SUM OF THE AREA RECTANGLES. THIS IS ONE OF THE DIFFERENT TECHNIQUES OF FINDING THE AREA OF A REGIO CURVE.

ACTIVITY 5.6



1 LET $x_0, x_1, x_2, \ldots, x_n \in [a, b]$ WITH $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.

THE FINITE SET $P = \{x_0, x_1, x_2, \ldots, x_n\}$ IS SAID TO BE A partition OF $[a, b]$.

FOR INSTANCE, $\left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ IS A PARTITION OF [0, 1].

FIND AT LEAST THREE DIFFERENT PARTITIONS OF [0, 1]

2 THE n - SUB INTERVALS IN WHICH THE PARTITION P DIVIDES $[a, b]$ ARE:

$[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$.

THE LENGTH OF THE k^{th} SUB INTERVAL $[x_{k-1}, x_k]$ IS $x_k - x_{k-1}$.

A DIVIDE [0, 1] INTO 5 - SUB INTERVALS OF EQUAL LENGTHS

B DIVIDE [3, 5] INTO 10 - SUB INTERVALS OF EQUAL LENGTHS.

C DIVIDE [0, 1] INTO n - SUB INTERVALS OF EQUAL LENGTHS.

3 CONSIDER THE AREA UNDER THE LINE $y = x + 1$ FROM $x = 0$ TO $x = 1$. DIVIDE THE INTERVAL [0, 1] INTO n - SUB INTERVALS EACH OF $\frac{1}{n}$. FIGURE 5.5 BELOW SHOWS A SKETCH OF THE INSCRIBED RECTANGLE.

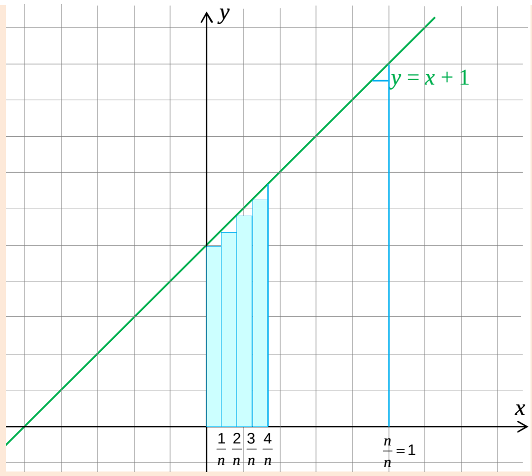


Figure 5.5

i FIND THE SUM OF THE AREAS OF THE RECTANGLES WHEN

a $n = 3$ **b** $n = 5$ **c** $n = 10$

ii FIND THE LIMITING VALUE OF THE SUM OF THE AREAS OF THE RECTANGLES, AS N $\to \infty$

4 REPEAT PROBLEM 3, IF THE RECTANGLES ARE CIRCUMSCRIBED INSTEAD OF BEING INSC

LOOK AT FIGURE 5.6.

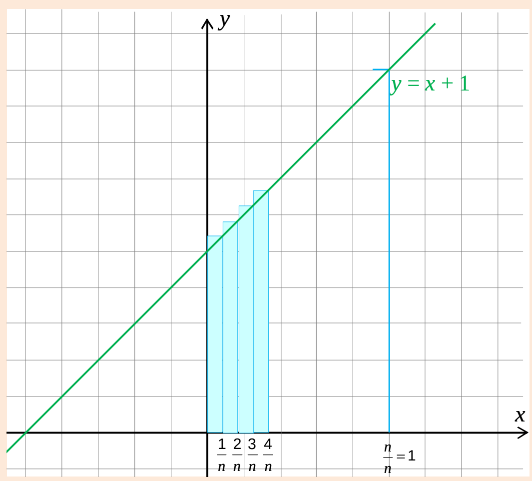


Figure 5.6

USING THE CONCEPT DEVELOPED IN THE ACTIVITY, CONSIDER A FUNCTION f WHICH IS NON-NEGATIVE AND CONTINUOUS ON $[a, b]$. THEN THE AREA UNDER THE CURVE $y = f(x)$ AND THE x-AXIS BETWEEN THE LINES $x = a$ AND $x = b$ IS CALCULATED AS FOLLOWS.

DIVIDE THE INTERVAL $[a, b]$ INTO n SUB INTERVALS

$[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$ EACH OF LENGTH $\Delta x = \dfrac{b-a}{n}$

LET n RECTANGLES EACH OF WIDTH $\dfrac{b-a}{n}$ BE INSCRIBED IN THE REGION AS SHOWN IN FIGURE 5.7

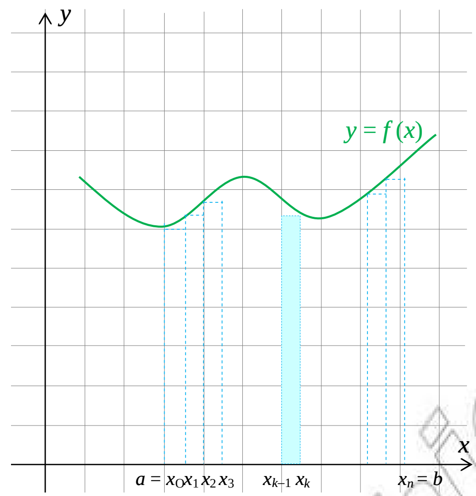


Figure 5.7

LET $z_k \in [x_{k-1}, x_k]$ SUCH THAT $f(z_k)$ IS THE HEIGHT OF k^{TH} RECTANGLE.

LET ΔA_k BE THE AREA OF k^{TH} RECTANGLE.

THEN, $\Delta A_k = \left(\frac{b-a}{n}\right) f(z_k)$

LET A BE THE SUM OF THE n RECTANGLES.

THEN, $A = \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} \frac{b-a}{n} f(z_k) = \left(\frac{b-a}{n}\right) \sum_{k=1}^{n} f(z_k)$

THE AREA A OF THE REGION IS THE LIMITING VALUE OF A_n WHEN $n \to \infty$.

I.E. $A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(z_k)$

Definition 5.3

1. THE SUM $\sum_{k=1}^{n} f(z_k)\Delta x$ IS SAID TO BE THE INTEGRAL SUM OF THE FUNCTION f ON THE INTERVAL $[a, b]$.
2. IF $\lim_{n\to\infty} \sum_{k=1}^{n} f(z_k)\Delta x$ EXISTS AND IS EQUAL TO I, THEN I IS SAID TO BE THE DEFINITE INTEGRAL OF f OVER THE INTERVAL $[a, b]$ AND IS DENOTED BY $I = \int_a^b f(x)\,dx$. a AND b ARE SAID TO BE THE lower AND upper limits OF INTEGRATION, RESPECTIVELY.

Example 1 FIND THE AREA OF THE REGION ENCLOSED BY THE GRAPH OF $f(x) = x^2$ AND THE x-AXIS BETWEEN THE LINES $x = 0$ AND $x = 1$.

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SOLUTION

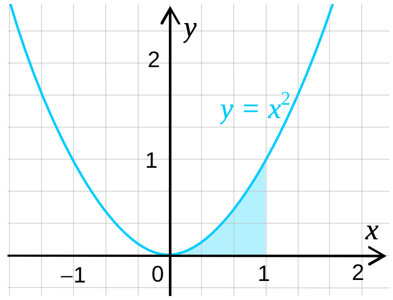

Figure 5.8

USING THE DEFINITION, CALCULATE THE AREA OF THE REGION AS FOLLOWS.

$$A=\int_a^b f(x)\,dx \Rightarrow \int_a^b x^2\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(z_k)\Delta x$$

WHERE $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ AND $z_k = \frac{k-1}{n} \Rightarrow f(z_k) = \left(\frac{k-1}{n}\right)^2$

$$\Rightarrow \sum_{k=1}^{n} f(z_k)\Delta x = \sum_{k=1}^{n}\left(\frac{k-1}{n}\right)^2\left(\frac{1}{n}\right) = \frac{1}{n^3}\sum_{k=1}^{n}(k-1)^2$$

$$= \frac{1}{n^3}\left[0+1+2^2+3^2+\ .\ .\ .\ +(n-1)^2\right]$$

$$= \frac{1}{n^3}\frac{(n-1)(n)\big(2(n-1)+1\big)}{6} = \frac{1}{6n^3}\left[2n^3-3n^2+n\right]$$

$$= \frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}$$

$$\Rightarrow A = \int_a^b x^2\,dx = \lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}\right) = \frac{1}{3}$$

Theorem 5.2 Estimate of the definite integral

IF THE FUNCTION f IS CONTINUOUS ON $[a, b]$, THEN $\lim_{n\to\infty}\sum f(z_i)\Delta x$ EXISTS.

THAT IS, THE DEFINITE INTEGRAL $\int_a^b f(x)\,dx$ EXISTS.

Example 2 SHOW THAT $\int_0^{\frac{\pi}{2}} \sin x\,dx$ EXISTS.

Solution $f(x) = \sin x$ IS CONTINUOUS ON $\left[0, \frac{\pi}{2}\right]$.

THUS, BY THE ABOVE THEOREM, THE DEFINITE INTEGRAL EXISTS.

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Example 3 SHOW THAT $\int_{-1}^{2}\frac{1}{x}\,dx$ DOESN'T EXIST.

Solution $f(x) = \frac{1}{x}$ IS DISCONTINUOUS AT $x = 0$.

$\Rightarrow$ f IS NOT CONTINUOUS ON $[-1, 2]$ $\Rightarrow$ $\int_{-1}^{2}\frac{1}{x}\,dx$ DOESN'T EXIST.

Exercise 5.9

1. ESTIMATE THE AREA OF THE REGION BOUNDED BY THE GRAPH OF $f(x) = x^3$ AND THE x-AXIS BETWEEN THE LINES $x = 1$ AND $x = 2$, BY DIVIDING THE INTERVAL [1, 2] INTO
   - A 5 - SUB INTERVALS
   - B 10 - SUB INTERVALS
   - C n - SUB INTERVAL, OF EQUAL LENGTHS.
2. USING THE DEFINITION OF AREA UNDER A CURVE, DETERMINE THE AREA OF THE REGION ENCLOSED BY THE CURVE $y = f(x)$ AND THE x-AXIS BETWEEN THE LINES $x = a$ AND $x = b$, WHEN
   - A $f(x) = 3x - 1$; $a = 1$, $b = 3$
   - B $f(x) = x^3$; $a = 1$, $b = 2$
   - C $f(x) = x^2 - 4x$; $a = 0$, $b = 4$
   - D $f(x) = x^2 - 2x + 1$; $a = 0$, $b = 2$,
3. IN EACH OF THE FOLLOWING DETERMINE WHETHER OR NOT THE INTEGRAL OF THE FUNCTION EXISTS ON THE GIVEN INTERVAL.
   - A $f(x) = \tan x$; $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
   - B $f(x) = \cos x$; $\left[-\pi, \frac{3}{2}\right]$
   - C $f(x) = |x|$; $[-3, 1]$
   - D $f(x) = \frac{x}{x^2 - 1}$; $[-2, 2]$
   - E $f(x) = \frac{2x}{x^2 - 9}$; $[-4, 4]$
   - F $f(x) = \frac{x + 1}{x^2 - 4}$; $[-1, 1]$
4. USING THE DEFINITION, EVALUATE EACH OF THE FOLLOWING DEFINITE INTEGRALS.
   - A $\int_{1}^{5} 4\,dx$
   - B $\int_{0}^{3} x\,dx$
   - C $\int_{-1}^{1}\left(x^2 + 1\right)dx$
   - D $\int_{-1}^{1}\frac{1}{x^2}\,dx$
   - E $\int_{-1}^{1} x^3\,dx$
5. EVALUATE $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\,dx$. WHAT IS THIS DEFINITE INTEGRAL REPRESENTING?
6. USING THE FACT $\sum_{x=1}^{n} x^2 = \sum_{s=1}^{n} s^2$ SHOW THAT $\int_a^b f(x)\,dx = \int_a^b f(s)\,ds$
7. IN FIGURE 5.9, THE AREA A OF THE REGION IS EQUAL TO

   $$A = A_1 + A_2$$

   USE THIS FACT TO EXPLAIN $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

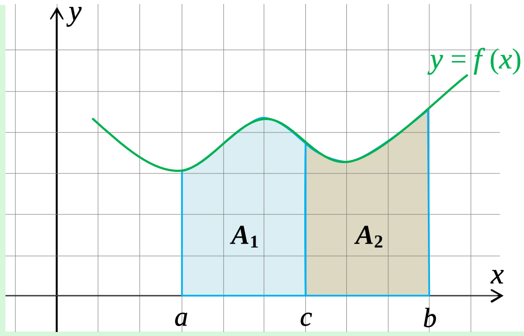

Figure 5.9

5.3.2 Fundamental Theorem of Calculus

FUNDAMENTAL THEOREM OF CALCULUS IS THE STATEMENT WHICH ASSERTS THAT DIFFERE INTEGRATION ARE INVERSE OPERATIONS OF EACH OTHER. TO UNDERSTAND THIS, LET f BE CONTINUOUS ON $[a, b]$. IF YOU FIRST INTEGRATE f AND THEN DIFFERENTIATE THE RESULT YOU CAN RETRIEVE BACK THE ORIGINAL FUNCTION f. THE NEXT THEOREM ALLOWS YOU TO EVALUATE THE DEFINITE INTEGRAL BY USING THE ANTI DERIVATIVE OF THE FUNCTION TO BE INTEGRATED.

Theorem 5.3 Fundamental theorem of calculus

IF f IS CONTINUOUS ON THE CLOSED INTERVAL $[a, b]$ AND F IS AN ANTI DERIVATIVE (OR INDEFINITE INTEGRAL) OF f,

THAT IS, $F'(x) = f(x)$ FOR ALL $x \in [a, b]$, THEN $\int_a^b f(x)\,dx = F(b) - F(a)$

Example 4 EVALUATE $\int_1^4 x\,dx$

Solution THIS VALUE IS CALCULATED USING THE DEFINITION OF DEFINITE INTEGRALS. HERE YOU USE THE FUNDAMENTAL THEOREM OF CALCULUS.

THE INDEFINITE INTEGRAL,

$$F(x) = \int x\,dx = \frac{x^2}{2} + c \Rightarrow \int_1^4 x\,dx = F(4) - F(1) = \left(\frac{4^2}{2} + c\right) - \left(\frac{1^2}{2} + c\right) = \frac{15}{2}$$

OBSERVE THAT EVALUATING THE DEFINITE INTEGRAL USING THE INTEGRAL SUM IS LENGTH COMPLICATED AS COMPARED TO USING THE FUNDAMENTAL THEOREM OF CALCULUS.

✍Note:

IN EVALUATING $F(b) - F(a)$, THE CONSTANT OF INTEGRATION CANCELS OUT.

THEREFORE, YOU WRITE $F(x)\Big|_a^b$ TO MEAN $F(b) - F(a)$

Example 5 EVALUATE $\int_1^3 (x^3 + x + 1)\,dx$

SOLUTION $\int_1^3 (x^3+x+1)\,dx = \left.\frac{x^4}{4}+\frac{x^2}{2}+x\right|_1^3 = \left(\frac{3^4}{4}+\frac{3^2}{2}+3\right)-\left(\frac{1}{4}+\frac{1}{2}+1\right)$

$$= \frac{81}{4}+\frac{9}{2}+3-\frac{1}{4}-\frac{1}{2}-1 = \frac{80}{4}+\frac{8}{2}+2 = 26$$

Example 6 EVALUATE $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x\,dx$

Solution $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x\,dx = -\cos x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = -\left[\cos\frac{\pi}{3} - \cos\frac{\pi}{6}\right] = -\left[\frac{1}{2}-\frac{\sqrt{3}}{2}\right] = \frac{\sqrt{3}-1}{2}$

Example 7 FIND THE AREA OF THE REGION BOUNDED BY THE ARC OF THE SINE FUNCTION BETWEEN $x = 0$ AND $x = \pi$.

Solution THE AREA A OF THIS REGION IDENTIFIED TO BE THE VALUE OF THE DEFINITE INTEGRAL $\int_0^{\pi} \sin x\,dx$.

$$\Rightarrow A = \int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} = -[\cos\pi - \cos 0] = -[-1-1] = 2$$

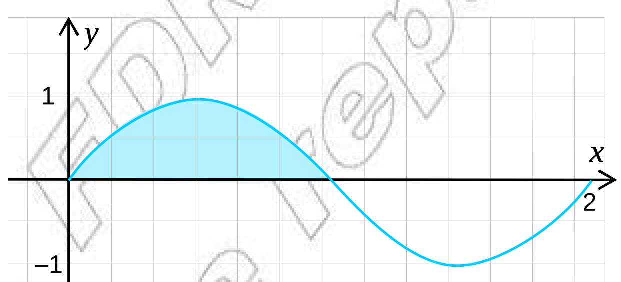

Figure 5.10

Example 8 EVALUATE $\int_{-1}^{0} e^x\,dx$

Solution $\int_{-1}^{0} e^x\,dx = e^x\Big|_{-1}^{0} = e^0 - e^{-1} = 1 - \frac{1}{e} = \frac{e-1}{e}$

Example 9 EVALUATE $\int_1^e \ln x\,dx$

Solution $\int_1^e \ln x\,dx = (x\ln x - x)\Big|_1^e = (e\ln e - e) - (1\times\ln 1 - 1) = (e - e) - (0-1) = 1$

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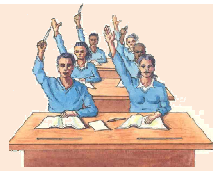
Properties of the definite integral

ACTIVITY 5.7



LET $f(x) = x^2$ AND $g(x) = 1 - \frac{1}{x}$.

1 EVALUATE EACH OF THE FOLLOWING DEFINITE INTEGRALS.

A $\int_1^3 (f(x) + g(x))\,dx$ **B** $\int_{-2}^3 f(x)\,dx$

C $\int_3^1 f(x)\,dx + \int_1^3 f(x)\,dx$ **D** $\int_3^3 f(x)\,dx$

E $4\int_{-2}^3 f(x)\,dx$ **F** $\int_1^4 g(x)\,dx + \int_4^{10} g(x)\,dx - \int_1^{10} g(x)\,dx$

2 LET f AND g BE CONTINUOUS FUNCTIONS ON THE CLOSED INTERVAL $[a, b]$ AND $k \in \mathbb{R}$

A EVALUATE $\int_a^a f(x)\,dx$ **B** EXPRESS $\int_b^a f(x)\,dx$ IN TERMS OF $\int_a^b f(x)\,dx$

C IN THE INDEFINITE INTEGRAL YOU LEARNED THAT

$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$ AND $\int kf(x)\,dx = k\int f(x)\,dx$.

DOES THIS PROPERTY HOLD TRUE FOR DEFINITE INTEGRALS? JUSTIFY YOUR ANSWER BY PRODUCING EXAMPLES.

D YOU ALSO LEARNED THAT $\sum_{i=1}^{n} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{n} a_i$ FOR $1 \le k < n$. DOES THE EQUALITY

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ FOR $a \le c < b$ HOLD TRUE?

E IN DIFFERENTIAL CALCULUS YOU SAW THAT $\frac{d}{dx}(f(x)g(x)) \ne \frac{d}{dx}f(x)\frac{d}{dx}g(x)$.

GIVE AN EXAMPLE TO SHOW THAT $\int_a^b f(x)g(x)\,dx \ne \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$.

SHOW THAT $\int_a^b \frac{f(x)}{g(x)}\,dx \ne \frac{\int_a^b f(x)\,dx}{\int_a^b g(x)\,dx}$ BY PRODUCING EXAMPLES.

Properties of the definite Integral

IF f AND g ARE CONTINUOUS ON $[a, b]$, $k \in \mathbb{R}$ AND $c \in [a, b]$ THEN

1 $\int_a^a f(x)\,dx = 0$

2 $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

3 $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

4 $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

5 $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

Example 10 EVALUATE EACH OF THE FOLLOWING INTEGRALS USING THE PROPERTIES AB

A $\int_3^3 (x^3+1)\,dx$ **B** $\int_{\frac{}{4}} \sin x\,dx$ **C** $\int_1^2 \left(x-\frac{1}{x^2}\right)^2 dx$

D $\int_1^{\sqrt{2}} \frac{x}{x^2+1}dx + \int_{\sqrt{2}}^{5} \frac{x}{x^2+1}dx$ **E** $\int_{-1}^{1} e^{\;+3x}\,dx$

Solution

A BY PROPERTY 1, $\int_3^3 \left(x^3+1\right) dx = 0$

B BY PROPERTY 2,

$$\int_{\frac{}{4}} \sin x\,dx = -\int^{\frac{}{4}} \sin x\,dx = \cos x\Big|^{\frac{}{4}} = \cos\frac{}{4} \quad ($$

$$= \frac{\sqrt{2}}{2} - (-1) = \frac{\sqrt{2}}{2} + 1$$

C BY PROPERTY 3 AND PROPERTY 5,

$$\int_1^2 \left(x-\frac{1}{x^2}\right)^2 dx = \int_1^2 \left(x^2 - \frac{2}{x} + \frac{1}{x^4}\right) dx$$

$$= \int_1^2 x^2\,dx - 2\int_1^2 \frac{1}{x}dx + \int_1^2 \frac{1}{x^4}dx = \frac{x^3}{3}\Big|_1^2 - 2\ln|x|\Big|_1^2 - \frac{1}{3x^3}\Big|_1^2$$

$$= \left(\frac{8}{3}-\frac{1}{3}\right) - 2[\ln 2 - \ln 1] - \left[\frac{1}{3(2^3)} - \frac{1}{3}\right]$$

$$= \frac{7}{3} - 2\ln 2 + \frac{7}{24} = \frac{21}{8} - 2\ln$$

D BY PROPERTY 4,

$$\int_1^{\sqrt{2}} \frac{x}{x^2+1}dx + \int_{\sqrt{2}}^{5} \frac{x}{x^2+1}dx = \int_1^5 \frac{x}{x^2+1}dx$$

$$= \frac{1}{2}\ln\left|x^2+1\right|\Big|_1^5$$

$$= \frac{1}{2}(\ln 26 - \ln 2) = \frac{1}{2} \quad \ln$$

E $\int_{-1}^{1} e^{\;+3x}\,dx = \int_{-1}^{1} e\; e^{3x}\,dx\; e\;.\frac{e^{3x}}{3}\Big|_{-1}^{1}$

$$\int_{-1}^{1} e^{\;+3x}\,dx = \int_{-1}^{1} e\; e^{3x}\,dx = e\int_{-1}^{1} e^{3x}\,dx = e\; e^{3x}\Big|_{-1}^{1} = e\left(\frac{e^3}{3} - \frac{e^{-3}}{3}\right) = \frac{e^{\;-3}}{3}\left(e^6-1\right)$$

$$= e\left(\frac{e^3}{3} - \frac{e^3}{3}\right) = \frac{e^{\;-3}}{3}\,(e^6-1).$$

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Change of variable

IN EVALUATING THE INDEFINITE INTEGRAL $\int f(x)\,dx = F(x)$, THE METHODS YOU HAVE BEEN USING ARE: SUBSTITUTION, PARTIAL FRACTIONS AND INTEGRATION BY PARTS.

IN THE SUBSTITUTION METHOD, THE COMPOSITE FUNCTION fog IS THE ANTIDERIVATIVE OF $(fog).g'$.

$$\Rightarrow \int_a^b f\left(g(x)\right).g'(x)\,dx = F\left(g(b)\right) - F\left(g(a)\right)$$

TO EVALUATE THE DEFINITE INTEGRAL BY THE METHOD OF SUBSTITUTION, YOU TRAN INTEGRAND AS WELL AS THE LIMITS OF INTEGRATION.

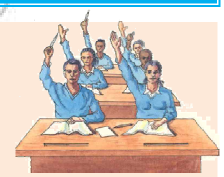
FOR THIS PROCESS YOU HAVE THE FOLLOWING THEOREM.

Theorem 5.4 Change of variables

IF THE FUNCTION f IS CONTINUOUS ON A CLOSED INTERVAL $[c, d]$, THE SUBSTITUTION FUNCTION $u = g(x)$ IS DIFFERENTIABLE ON $[a, b]$ WITH $g(a) = c$ AND $g(b) = d$, THEN

$$\int_a^b f\left(g(x)\right).g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

ACTIVITY 5.8



1 IF $-1 \le x \le 2$, FIND THE INTERVALS OF VALUES FOR u, WHEN

A $u = 3x - 1$ B $u = \sqrt{x+1}$ C $u = e^{x^2+1}$

D $u = 1 - x\sqrt{x^2-1}$ E $u = x^2 - x - 2$

2 IF $\int_a^b \frac{2x}{\sqrt{x^2+4}}\,dx = \int_c^d \frac{du}{\sqrt{u}}$, FIND THE VALUES OF c AND d IN TERMS OF a AND b.

3 EVALUATE $\int_1^2 (2x+1)\sqrt{x^2+x-2}\,dx$ USING THE METHOD OF SUBSTITUTION.

Example 11 EVALUATE THE INTEGRAL $\int_1^2 xe^{(x^2)}\,dx$

Solution USING INTEGRATION BY SUBSTITUTION,

$$u = x^2, \frac{du}{dx} = 2x \Rightarrow \frac{1}{2}\,du = x\,dx$$

AS x VARIES FROM 1 TO 2, $u = g(x)$ VARIES FROM $g(1) = 1$ TO $g(2) = 2^2 = 4$.

$$\int_1^2 xe^{x^2}\,dx = \frac{1}{2}\int_1^4 e^u\,du = \frac{1}{2}e^u\Big|_1^4 = \frac{1}{2}\left(e^4 - e\right)$$

Example 12 EVALUATE THE INTEGRAL $\int_{-3}^{1} x\sqrt{2x^2+5}\,dx$.

Solution HERE, $u = g(x) = 2x^2 + 5$, $g(-3) = 2(-3)^2 + 5 = 23$,

$g(1) = 2(1)^2 + 5 = 7$

$$\frac{du}{dx} = \frac{d}{dx}\left(2x^2+5\right) = 4x \Rightarrow \frac{1}{4}du = x\,dx$$

$$\int_{-3}^{1} x\sqrt{2x^2+5}\,dx = \frac{1}{4}\int_{23}^{7}\sqrt{u}\,du = \frac{1}{4}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_{23}^{7} = \frac{1}{6}\left(7\sqrt{7} - 23\sqrt{23}\right).$$

Example 13 EVALUATE $\int_0^{\frac{\pi}{3}} \cos^3 x \sin x\,dx$

Solution THE DERIVATIVE OF $\cos x$ IS $-\sin x$ WHICH IS A FACTOR OF THE INTEGRAND.

HENCE, $u = g(x) = \cos x$.

$\Rightarrow -du = \sin x\,dx$

$$\int_0^{\frac{\pi}{3}} \cos^3 x \sin x\,dx = -\int_{g(0)}^{g\left(\frac{\pi}{3}\right)} u^3\,du = -\int_1^{\frac{1}{2}} u^3\,du = -\frac{u^4}{4}\Bigg|_1^{\frac{1}{2}} = -\left(\frac{1}{64} - \frac{1}{4}\right) = \frac{15}{64}$$

Exercise 5.10

IN EXERCISES 1–15 EVALUATE EACH OF THE FOLLOWING DEFINITE INTEGRALS USING THE FUNDAMENTAL THEOREM OF CALCULUS. IN THE EXERCISES, $a, b \in \mathbb{R}$ AND $n \in \mathbb{N}$.

1. $\int_{-1}^{4} 3\,dx$
2. $\int_a^b dx$
3. $\int_{-1}^{5} -x\,dx$
4. $\int_1^2 x^5\,dx$
5. $\int_a^b x^n\,dx$
6. $\int_2^1 \left(x^3 + 5x^2 - 1\right)dx$
7. $\int_1^2 \sqrt{2x-1}\,dx$
8. $\int_0^1 2^{3-x}\,dx$
9. $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$
10. $\int_2^3 \frac{1}{x}\,dx$
11. $\int_0^1 \frac{9}{4x+1}\,dx$
12. $\int_0^{\sqrt{\pi}} \sin\left(x^2 + 3\right)dx$
13. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin x\,dx$

4 $\int_e^{e^3} x\ln x\,dx$

15. $\int_1^3 \frac{1}{2x^2+x-1}\,dx$

IN EXERCISES 16 – 25, EVALUATE EACH OF THE FOLLOWING DEFINITE INTEGRALS USING CHANGE OF VARIABLES.

16. $\int_{-1}^{1} \frac{2x+3}{\left(x^3+3x+4\right)^6}\,dx$
17. $\int_{-1}^{\frac{1}{2}} (4x+3)^{10}\,dx$
18. $\int_{\sqrt{2}}^{3} x\sqrt{x^2+7}\,dx$

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19 $\int_1^2 x^2\left(x^3-3\right)^5 dx$ **20** $\int_4^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$ **21** $\int_0^2 (x-2)\sqrt{x+1}\, dx$

22 $\int_{-1}^0 \frac{x+4}{3x+1} dx$ **23** $\int_{-4}^3 \frac{x+1}{x^2-x-6} dx$ **24** $\int_{-1}^1 \frac{3t^2-1}{e^{t^3-t}} dt$

25 $\int_{-1}^1 \frac{e^x}{1+e^x}\, dx$

26 YOU KNOW THAT $\int \frac{1}{x^2} dx = -\frac{1}{x} + c$. IS $f(x)$ INTEGRABLE ON $[-2, 2]$? IF SO, FIND $\int_{-2}^{2} \frac{1}{x^2}\, dx$.

27 LET f BE AN EVEN FUNCTION WHICH IS CONTINUOUS ON $[-a, a]$ FOR ANY REAL NUMBER a. THEN, SHOW THAT $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$. USE $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x\, dx$ TO VERIFY YOUR WORK

28 IF f IS AN ODD FUNCTION THAT IS CONTINUOUS ON $[-a, a]$, FOR ANY REAL NUMBER a, THEN SHOW THAT $\int_{-a}^{a} f(x)\, dx = 0$

VERIFY YOUR CONCLUSION BY COMPUTING THE FOLLOWING INTEGRALS

A $\int_{-3}^{3}\left(x^3+x\right)dx$ **B** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\, dx$ **C** $\int_{-1}^{1} \frac{x}{x^2+1}\, dx$

5.4 APPLICATIONS OF INTEGRAL CALCULUS

IN THIS SECTION, YOU SHALL SEE SOME OF THE MATHEMATICAL AND PHYSICAL APPLICA INTEGRAL CALCULUS. IN THE MATHEMATICAL APPLICATION YOU CALCULATE THE AREA BOUNDED BY CURVES OF CONTINUOUS FUNCTIONS DEFINED ON THE INTERVAL $[a, b]$ AND ALSO THE VOLUME OF A SOLID OF REVOLUTION.

IN THE PHYSICAL APPLICATIONS, YOU CALCULATE THE WORK DONE BY A VARIABLE FORC STRAIGHT LINE, ACCELERATION, VELOCITY AND DISPLACEMENT.

5.4.1 The Area Between Two Curves

YOU CALCULATED THE AREA OF SOME REGIONS UNDER THE GRAPHS OF f A NON-NEGATIVE ON $[a, b]$, WHEN THE DEFINITE INTEGRAL $\int_a^b f(x)dx$ WAS DEFINED. HOWEVER THE FOCUS WAS TO EVALUATE THE INTEGRAL RATHER THAN TO CALCULATE AREA. HERE, YOU USE THIS CONC ORDER TO DETERMINE THE AREA OF A REGION WHOSE UPPER AND LOWER BOUNDARIES AR CONTINUOUS FUNCTIONS ON A GIVEN CLOSED INTERVAL $[a, b]$.

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ACTIVITY 5.9

1 USING THE DEFINITION OF THE DEFINITE INTEGRAL, CALCULATE THE AREA OF THE REGION BOUNDED BY THE GRAPH OF

A $y = x$ AND THE x-AXIS BETWEEN $x = 0$ AND $x = 1$.

B $y = x^2 + 1$ AND THE x-AXIS BETWEEN $x = 0$ AND $x = 1$.

2 USING THE RESULT FROM PROBLEM 1, AND YOUR KNOWLEDGE OF THE AREA OF A SHADED PART, FIND THE AREA OF THE REGION BOUNDED BY THE GRAPHS OF $f(x) = x^2 + 1$ AND $g(x) = x$ BETWEEN $x = 0$ AND $x = 1$.

WE EXTEND THE PROBLEMS OF ACTIVITY 5.9 TO AN ARBITRARY REGION ENCLOSED BY THE GRAPHS OF CONTINUOUS FUNCTIONS.

Example 1 FIND THE AREA OF THE REGION BOUNDED BY THE GRAPH OF THE FUNCTION $f(x) = x^2 - 3x + 2$ AND THE x-AXIS BETWEEN $x = 0$ AND $x = 3$.

Solution LOOK AT THE GRAPH OF f BETWEEN $x = 0$ AND $x = 3$.

LET A_1, A_2 AND A_3 BE THE AREAS OF THE PARTS OF THE REGION BETWEEN $x = 0$ AND $x = 1$, $x = 1$ AND $x = 2$ AND $x = 2$ AND $x = 3$, RESPECTIVELY.

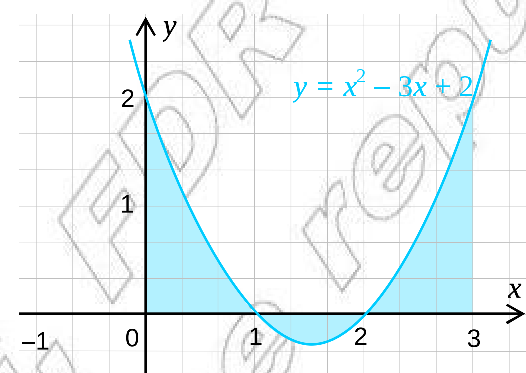

Figure 5.11

THE PART OF THE REGION BETWEEN $x = 1$ AND $x = 2$ IS BELOW THE x-AXIS.

$$\Rightarrow A_2 = -\int_1^2 \left(x^2 - 3x + 2\right) dx = -\left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\Bigg|_1^2 = 4 - \frac{23}{6} = \frac{1}{6}$$

WHEREAS, $A_1 = \int_0^1 (x^2 - 3x + 2)\, dx = \left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\Bigg|_0^1 = \frac{5}{6}$ AND

$$A_3 = \int_2^3 \left(x^2 - 3x + 2\right) dx = \left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\Bigg|_2^3 = \frac{5}{6}$$

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THEREFORE, THE AREA A OF THE REGION IS

$$A = A_1 + A_2 + A_3 = \frac{11}{6}$$

WHAT WOULD HAVE HAPPENED, IF YOU HAD SIMPLY TRIED TO CALCULATE A AS

$$A = \int_0^3 \left(x^2 - 3x + 2\right) dx?$$

Example 2 FIND THE AREA OF THE REGION ENCLOSED BY THE GRAPH OF $f(x) = \sin x$ AND THE x-AXIS BETWEEN $x = -\frac{\pi}{2}$ AND $x = 2$.

Solution

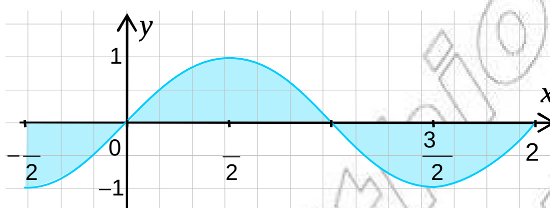

Figure 5.12

FROM THE GRAPH YOU HAVE THE AREA A OF THE REGION

$$A = -\int_{-\frac{}{2}}^{0} \sin x\, dx + \int_0 \sin x\, dx - \int^{2} \sin x\, dx$$

$$= -\cos x\Big|_{-\frac{}{2}}^{0} - \cos x\Big|_0 + \cos x\Big|^{2} = \cos x\Big|_{-\frac{}{2}}^{0} - \cos x\Big|^{0} + \cos x\Big|^{2}$$

$= \cos 0 + (\cos 0 - \cos) + (\cos (2) - \cos ())$

$= 1 + 1 - (-1) + 1 - (-1) = 5$

Example 3 FIND THE AREA OF THE REGION BOUNDED BY THE GRAPH OF $f(x) = x^5$ AND THE x-AXIS BETWEEN $x = -1$ AND $x = 1$.

Solution

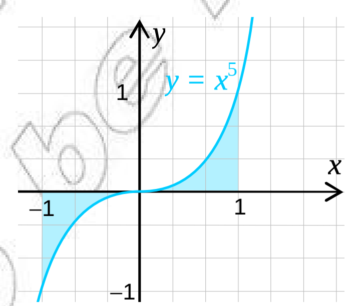

Figure 5.13

FROM THE SYMMETRY OF THE REGION, YOU HAVE THE AREA

$$A = 2\int_0^1 x^5\, dx = 2\left(\frac{x^6}{6}\right)\Bigg|_0^1 = \frac{1}{3}$$

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Example 4 FIND THE AREA OF THE REGION BOUNDED BY THE GRAPH OF $f(x) = x^3 - 2x^2 + x$ AND THE x-AXIS BETWEEN $x = -1$ AND $x = 2$.

Solution

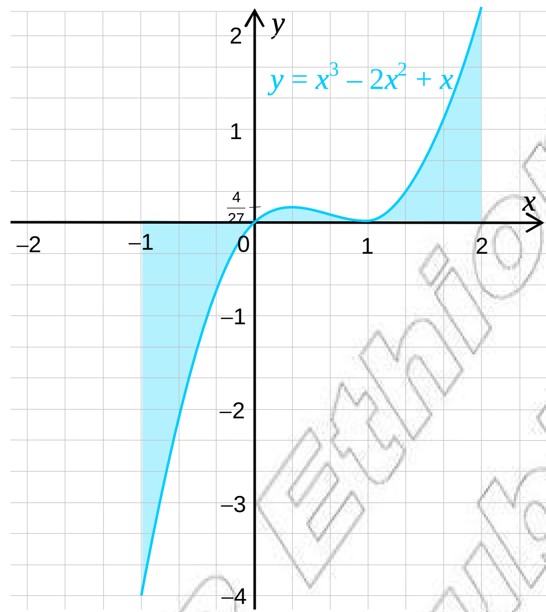

Figure 5.14

$f(x) = x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x - 1)^2$

LET THE AREA OF THE PART OF THE REGION UNDER THE x-AXIS BE A_1.

THEN, $A_3 = -\int_{-1}^{0}\left(x^3 - 2x^2 + x\right)dx = -\left(\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right)\Bigg|_{-1}^{0}$

$$= -\left[0 - \left(\frac{1}{4} + \frac{2}{3} + \frac{1}{2}\right)\right] = \frac{17}{12} \text{ SQUARE UNITS}$$

LET THE AREA OF THE REGION ABOVE THE x-AXIS BE A_2, THEN

$$A_2 = -\int_{0}^{2}\left(x^3 - 2x^2 + x\right)dx = -\left(\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right)\Bigg|_{0}^{2} = \left(\frac{16}{4} - \frac{16}{3} + 2\right) = \frac{2}{3}$$

$\Rightarrow$ THE AREA A OF THE REGION IS

$$A = A_1 + A_2 = \frac{17}{12} + \frac{2}{3} = \frac{25}{12}$$

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Example 5 Let $f(x) = \begin{cases} 2^x, & \text{if } x \le 1; \\ 1 + \dfrac{1}{x}, & \text{if } x > 1. \end{cases}$

Find the area of the region enclosed by the graph of f and the x-axis between $x = -1$ and $x = 2$.

Solution You first show that the function is continuous on $[-1, 2]$.

Look at the graph of f on $[-1, 2]$. The function is continuous on $[-1, 2]$.

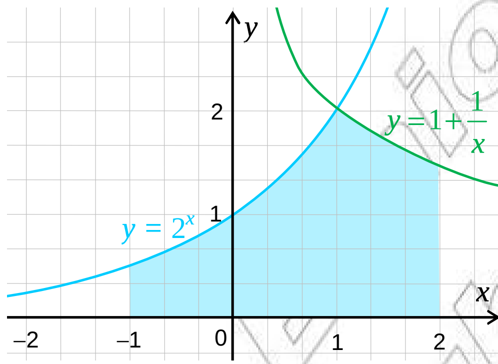


Figure 5.15

The upper part of the region is bounded by the graphs of two functions, $y = 2^x$ and $y = 1 + \frac{1}{x}$ intersecting at $x = 1$.

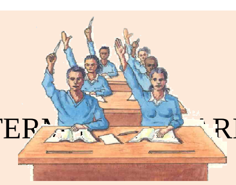
Let A_1 be the area of the region between the lines $x = -1$ and $x = 1$ and A_2 be the area of the region between the lines $x = 1$ and $x = 2$.

Then, $A_1 = \int_{-1}^{1} 2^x dx = \frac{2^x}{\ln 2}\Big|_{-1}^{1} = \frac{1}{\ln 2}\left(2 - \frac{1}{2}\right) = \frac{3}{2\ln 2} = \frac{3}{2\ln 2} = \frac{3}{\ln 4}$

$$A_2 = \int_1^2 \left(1 + \frac{1}{x}\right) dx = x + \ln|x|\Big|_1^2 = (2 + \ln 2) - (1 + \ln 1) = 1 + \ln 2$$

$\Rightarrow$ The area of the region $A = A_1 + A_2 = \frac{3}{\ln 4} + 1 + \ln 2$.

Activity 5.10



1 Using your knowledge of shaded area, determine the area of the region enclosed by the graphs of $f(x) = x^2 - 4$ and $g(x) = 1$ and the lines $x = 1$ and $x = 3$

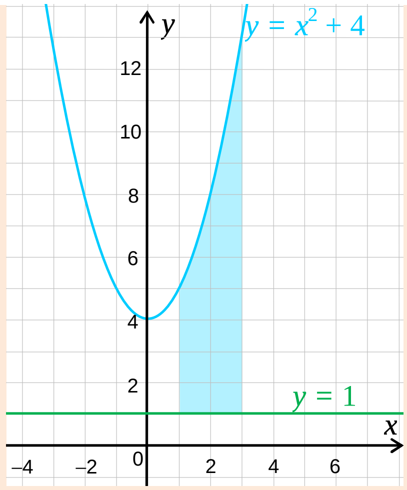


Figure 5.16

2 CONSIDER THE FOLLOWING REGION

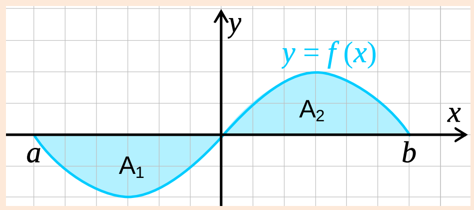


Figure 5.17

EXPRESS THE AREA A OF THE REGION IN TERMS OF THE INTEGRAL OF THE BOUNDARIES $y = f(x)$ AND $y = 0$.

Theorem 5.5

SUPPOSE f AND g ARE CONTINUOUS FUNCTIONS ON $[a, b]$ WITH $f(x) \geq g(x)$ ON $[a, b]$. THE AREA A BOUNDED BY THE CURVES $y = f(x)$ AND $y = g(x)$ BETWEEN THE LINES $x = a$ AND $x = b$ IS

$$A = \int_a^b \left(f(x) - g(x) \right) dx.$$

Example 6 FIND THE AREA OF THE REGION ENCLOSED BY THE CURVES $g(x) = x^2 - x - 6$ AND $f(x) = x - 3$.

Solution THE FIRST STEP IS TO DRAW THE GRAPHS OF BOTH FUNCTIONS USING THE SAME

YOU SOLVE THE EQUATION $f(x) = g(x)$ TO GET THE INTERSECTION POINTS OF THE GRAPHS.

$$x^2 - x - 6 = x - 3$$

$$\Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1) = 0 \Rightarrow x = 3 \text{ OR } x = -1$$

$f(x) \geq g(x)$ ON $[-1, 3]$

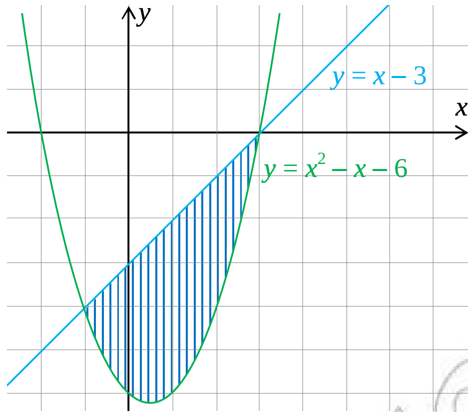


Figure 5.18

> **✍Note:**
>
> THE HEIGHT OF EACH INFINITESIMAL RECTANGLE WITHIN THE SHADED REGION IS EQUAL TO ($f(x) - g(x)$).

$\Rightarrow$ THE AREA OF THE REGION IS

$$A=\int_{-1}^{3}\left((x-3)-(x^2-x-6)\right)dx=\int_{-1}^{3}\left(-x^2+2x+3\right)dx=-\frac{x^3}{3}+x^2+3x\Bigg|_{-1}^{3}$$

$$=\frac{-27}{3}+9+9-\left(-\left(\frac{-1}{3}\right)+1+3(-1)\right)=9-\left(\frac{1}{3}+1-3\right)=\frac{32}{3}$$

Example 7 FIND THE AREA OF THE REGION ENCLOSED BY THE GRAPHS OF $f(x) = 4 - x^2$ AND $g(x) = 6 - x$ BETWEEN THE LINES $x = -2$ AND $x = 3$.

Solution THE FIRST STEP IS TO DRAW THE GRAPHS OF BOTH FUNCTIONS USING THE SAME COORDINATE AXES.

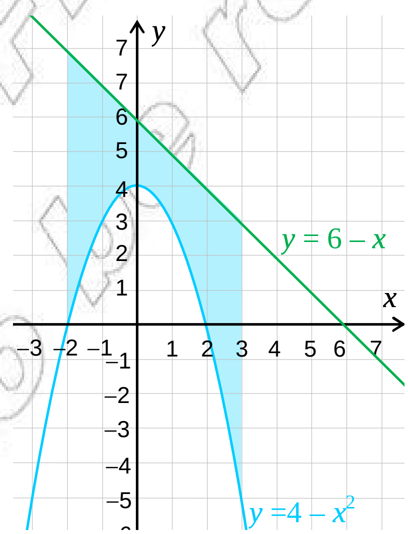


Figure 5.19

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$g(x) \geq f(x)$ ON $[-2, 3]$

$$\Rightarrow \text{THE AREA, } A = \int_{-2}^{3} \left((6-x)-(4-x^2)\right) dx = \int_{-2}^{3} (x^2 - x + 2)\, dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + 2x \Big|_{-2}^{3} = \frac{27}{3} - \frac{9}{2} + 6 - \left[\frac{-8}{3} - \frac{4}{2} - 4\right] = \frac{115}{6}$$

Example 8 FIND THE AREA OF THE REGION IN THE FIRST QUADRANT WHICH IS ENCLOSED BY THE y-AXIS AND THE CURVES OF $f(x) = \cos x$ AND $g(x) = \sin x$.

Solution LOOK AT THE GRAPHS OF BOTH FUNCTIONS.

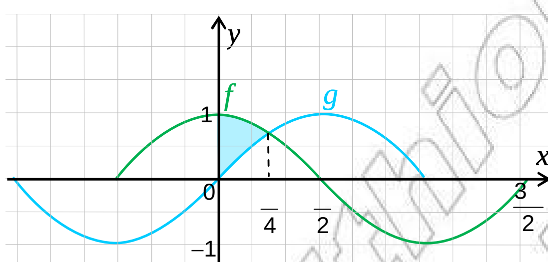

Figure 5.20

THE CURVES MEET AT $x = \frac{\pi}{4}$ AND $\cos x \geq \sin x$ ON $\left[0, \frac{\pi}{4}\right]$.

THEREFORE THE REQUIRED AREA IS

$$A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx = \left(\sin x - (-\cos x)\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} - (\sin 0 + \cos 0) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 = \sqrt{2} - 1$$

Example 9 FIND THE AREA ENCLOSED BY THE GRAPHS OF $f(x) = x^3 - x$ AND $g(x) = x^2 - 1$

Solution THE FIRST STEP IS TO DRAW BOTH GRAPHS.

SOLVE THE EQUATION

$x^3 - x = x^2 - 1$ TO FIND OUT THE INTERSECTION POINTS OF THE GRAPHS.

$x^3 - x = x^2 - 1$

$$\Rightarrow x^3 - x^2 - x + 1 = 0 \Rightarrow x^2(x-1) - (x-1) = 0 \Rightarrow (x^2 - 1)(x-1) = 0 \Rightarrow x = \pm 1$$

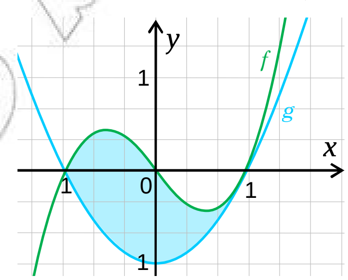

Figure 5.21

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THE REQUIRED AREA A IS $\int_{-1}^{1}\left(x^3 - x - \left(x^2 - 1\right)\right)dx = \int_{-1}^{1}\left(x^3 - x^2 - x + 1\right)dx$

$$= \left.\frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} + x\right|_{-1}^{1} = \frac{1}{4} - \frac{1}{3} - \frac{1}{2} + 1 - \left(\frac{1}{4} + \frac{1}{3} - \frac{1}{2} - 1\right) = \frac{4}{3}$$

Example 10 FIND THE AREA OF THE REGION ENCLOSED BY THE CURVES OF $f(x) = 2x^2 - x^3$ AND $g(x) = 4 - x^2$.

Solution THE FIRST STEP IS TO DETERMINE THE INTERSECTION POINTS OF THE GRAPHS AND THEN TO DRAW BOTH GRAPHS.

THUS, $2x^2 - x^3 = 4 - x^2 \Rightarrow x^3 - 3x^2 + 4 = 0$

USING THE RATIONAL ROOT TEST, THE ZEROS OF $x^3 - 3x^2 + 4$ ARE $x = -1$ AND $x = 2$.

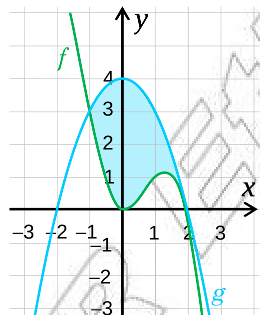

Figure 5.22

THE REQUIRED AREA A IS $\int_{-1}^{2}\left(4 - x^2\right) - \left(2x^2 - x^3\right)dx = \int_{-1}^{2}\left(x^3 - 3x^2 + 4\right)dx$

$$= \left.\frac{x^4}{4} - x^3 + 4x\right|_{-1}^{2} = \left(\frac{2^4}{4} - 2^3 + 4(2)\right) - \left(\frac{(-1)^4}{4} + 1 - 4\right)$$

$$= (4 - 8 + 8) - \left(\frac{1}{4} + 1 - 4\right) = \left(4 + 3 - \frac{1}{4}\right) = \frac{27}{4}$$

Example 11 FIND THE AREA ENCLOSED BY THE GRAPH OF $f(x) = |x|$ AND THE x-AXIS BETWEEN THE VERTICAL LINES $x = -4$ AND $x = 3$.

Solution DO YOU THINK THAT $\int_{-4}^{3}|x|\,dx$ EXISTS? *Explain!*

$f(x) = |x|$ IS CONTINUOUS ON $[-4, 3]$.

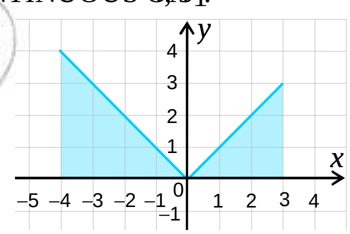

Figure 5.23

YOU KNOW THAT $f(x) = \begin{cases} x, & \text{IF } x \geq 0 \\ -x, & \text{IF } x < 0 \end{cases}$

THUS, THE AREA $A = \int_{-4}^{3} |x|\,dx = \int_{-4}^{0} |x|\,dx + \int_{0}^{3} |x|\,dx$

$$= \int_{-4}^{0} (-x)\,dx + \int_{0}^{3} x\,dx = -\frac{x^2}{2}\Bigg|_{-4}^{0} + \frac{x^2}{2}\Bigg|_{0}^{3} = -\left(0 - \frac{(-4)^2}{2}\right) + \left(\frac{3^2}{2} - 0\right) = \frac{25}{2}$$

Example 12 DETERMINE THE AREA OF THE REGION ENCLOSED BY THE GRAPHS OF $x = -y^2$ AND $x = 9 - 2y^2$.

Solution HERE THE CURVES ARE OPENING IN THE NEGATIVE x DIRECTION.

THE REGION IS SYMMETRICAL WITH RESPECT TO THE x-AXIS.

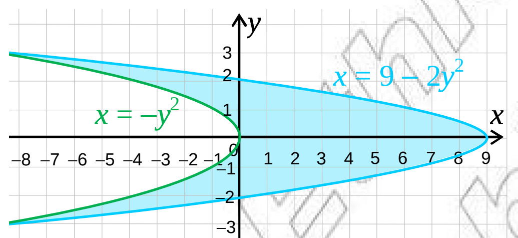

Figure 5.24

YOU SOLVE $-y^2 = 9 - 2y^2$ IN ORDER TO DETERMINE THE INTERSECTION POINTS OF THE GRAPHS.

THUS, $-y^2 = 9 - 2y^2 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$

THE REQUIRED AREA IS FOUND BY INTEGRATING WITH RESPECT TO y. [Can you see why?]

$$A = 2\int_{0}^{3} \left(\left(9 - 2y^2\right) + y^2\right) dy = 2\left(9y - \frac{y^3}{3}\right)\Bigg|_{0}^{3} = 2(27 - 9) = 36$$

Example 13 FIND THE AREA OF THE REGION ENCLOSED BY THE GRAPH OF $y = x^2 + 1$ AND THE LINE $y = 5$.

Solution FROM THE GRAPH, YOU SEE THAT THE LINE $y = 5$ CROSSES THE CURVE $y = x^2 + 1$ AT $x = \pm 2$.

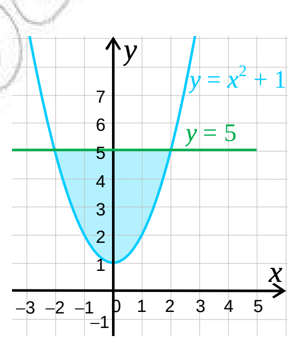

Figure 5.25

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$x^2 + 1 \le 5$ FOR ALL $x \in [-2, 2]$. THEREFORE, THE REQUIRED AREA IS

$$A = \int_{-2}^{2} \left(5 - \left(x^2 + 1\right)\right) dx = 4x - \frac{x^3}{3}\Bigg|_{-2}^{2} = 8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right) = \frac{32}{3}.$$

Exercise 5.11

1 FIND THE AREA OF THE REGION ENCLOSED BY THE GRAPH OF f AND THE x-AXIS FROM $x = a$ TO $x = b$ WHEN

- **A** $f(x) = x$; $x = -3$ AND $x = 2$.
- **B** $f(x) = 12 - 3x^2$; $x = -4$ AND $x = 3$.
- **C** $f(x) = x^3$; $x = -1$ AND $x = 1$.
- **D** $f(x) = 2^x$; $x = -1$ AND $x = 4$.
- **E** $f(x) = \ln x$; $x = \frac{1}{e}$ AND $x = e^2$.
- **F** $f(x) = \sin x$; $x = \frac{\pi}{4}$ AND $x = \frac{7}{4}\pi$.
- **G** $f(x) = \frac{1}{x}$; $x = \frac{1}{10}$ AND $x =$.
- **H** $f(x) = x^2 + 4$; $x = -1$ AND $x = \frac{1}{2}$.

2 FIND THE AREA OF THE REGION ENCLOSED BY THE GRAPHS OF

- **A** $f(x) = x^2$ AND $g(x) = \sqrt{x}$.
- **B** $f(x) = |x|$ AND $g(x) = x^2$.
- **C** $f(x) = 3x^2 - 4$ AND $g(x) = 2x^2$.
- **D** $f(x) = x^3 - 4x$ AND $g(x) = -3x^2$.

5.4.2 Volume of Revolution

OPENING PROBLEM

A HEMISPHERICAL BOWL OF RADIUS 5M CONTAINS SOME WATER. IF THE RADIUS OF THE SUR THE WATER IS 3M, WHAT IS THE VOLUME OF THE WATER?

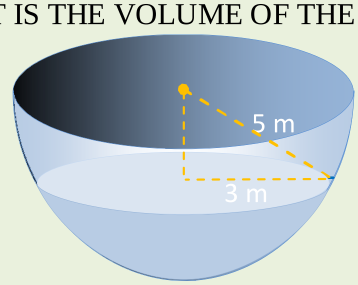
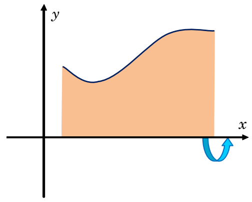


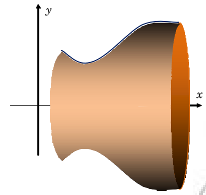
Figure 5.26

IN THIS SECTION, YOU WILL APPLY INTEGRAL CALCULUS TO DETERMINE THE VOLUME OF A SOLID BY CONSIDERING CROSS SECTIONS. IN THE STUDY OF PLANTS OR ANIMALS, VERY THIN CROSS S PREPARED BY SCIENTISTS. DURING EXAMINATION IN A TRANSMISSION ELECTRON MIC (TEM), THE ELECTRON BEAM CAN PENETRATE IF THE SLICED SPECIMEN IS EXTREMELY BECAUSE ONLY THE ELECTRONS THAT PASS THROUGH THE SPECIMEN ARE RECORDED.

SUPPOSE A REGION ROTATES ABOUT A STRAIGHT LINE AS SHOWN IN FIGURE 5.27A BELOW. THEN A SOLID FIGURE, CALLED A solid of revolution, WILL BE FORMED [SEE FIGURE 5.27B]



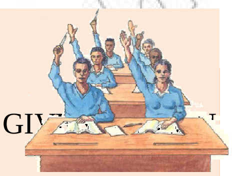
A *A region to be rotated*
(Two dimensional)



B *A solid of revolution*
(Three dimensional)

Figure 5.27

ACTIVITY 5.11



IDENTIFY THE TYPE OF THE SOLID SO FORMED WHEN THE GIVEN REGION ROTATES ABOUT THE x-AXIS:

1 THE REGION BOUNDED BY $y = 1$ AND THE x-AXIS BETWEEN $x = 0$ AND $x = 5$;

2 THE REGION BOUNDED BY $y = x$ AND THE x-AXIS BETWEEN $x = 0$ AND $x = 1$;

3 THE SEMICIRCLE $x^2 + y^2 = 1$; $0 \le y \le 1$ AND THE x-AXIS BETWEEN $x = -1$ AND $x = 1$;

4 THE REGION BETWEEN THE LINE $y = x + 1$ AND THE x-AXIS BETWEEN $x = 0$ AND $x = 3$;

5 THE REGION BOUNDED BY $y = \sqrt{4 - x^2}$ AND THE x-AXIS BETWEEN $x = -2$ AND $x = -1$.

FROM THE ABOVE ACTIVITY, YOU HAVE SEEN DIFFERENT SOLIDS FORMED BY ROTATING AN A LINE. IN GENERAL, A SOLID OF REVOLUTION IS A THREE DIMENSIONAL OBJECT FORMED BY AN AREA ABOUT A STRAIGHT LINE. THE NEXT TASK IS TO FIND THE VOLUME OF SUCH A SOLI

THE VOLUME OF A SOLID OF REVOLUTION IS SAID TO BE A volume of revolution. THE LINE ABOUT WHICH THE AREA ROTATES IS AN axis of symmetry.

NOW, CONSIDER THE FOLLOWING SOLID OF REVOLUTION GENERATED BY REVOLVING THE R BETWEEN THE CURVE $y = f(x)$ AND THE x-AXIS FROM $x = a$ TO $x = b$.

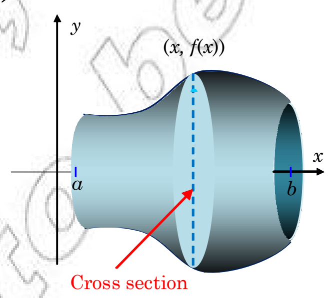


Figure 5.28

EVERY CROSS SECTION WHICH IS PERPENDICULAR TO THE x-AXIS IS A CIRCULAR REGION WITH RADIUS $r = f(x)$. THUS, THE AREA OF THE CROSS SECTION IS $r^2 = (f(x))^2$

How to determine the volume of a solid of revolution

DIVIDE THE SOLID OF REVOLUTION INTO n EQUALLY SPACED CROSS SECTIONS WHICH ARE PERPENDICULAR TO THE AXIS OF ROTATION [SEE FIGURE 5.29]

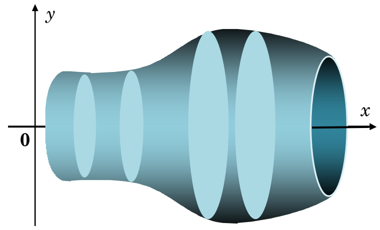


Figure 5.29

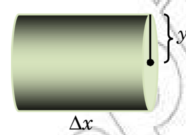


Figure 5.30

AS THE CUTS GET CLOSE ENOUGH, THEN EACH SECTION WILL APPROXIMATELY BE A CYLINDRICAL SOLID AS IN FIGURE 5.30

LET V_K BE THE VOLUME OF THE k^{th} SECTIONS, THEN

$$V_K = r^2 h, \text{ WHERE } r = f(x_k) \text{ AND } h = \Delta x$$

$$\Rightarrow V_k = (f(x_k))^2\,\Delta x$$

LET ΔV BE THE SUM OF THE VOLUMES OF THE n SECTIONS.

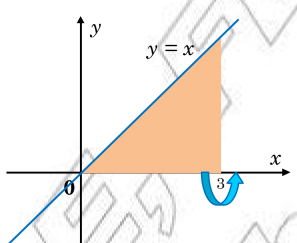
THEN, $\Delta V = \sum_{k=1}^{n} V_k$.

THE VOLUME V OF THE SOLID OF REVOLUTION IS

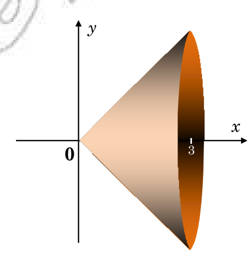
$$V = \lim_{\Delta x \to 0} \Delta V = \lim_{n\to\infty} \sum_{k=1}^{n} V_k = \lim_{n\to\infty} \sum_{k=1}^{n} \left(f(x_k)\right)^2 \Delta x = \int_a^b \left(f(x)\right)^2 dx$$

Example 14 FIND THE VOLUME GENERATED WHEN THE AREA BOUNDED BY THE LINE $y = x$ AND THE x-AXIS FROM $x = 0$ TO $x = 3$ IS ROTATED ABOUT THE x-AXIS.

SOLUTION



A *Rotating the region about the x – axis gives the solid as shown in the figure on the right. Using the definite integral the volume V is determined as follows*



B *The solid of revolution is a right circular cone with radius and height each 3 units long.*

Figure 5.31

$$V = \int_0^3 x^2\,dx = \left.\frac{x^3}{3}\right|_0^3 = 9$$

CHECK THAT YOU ARRIVE AT THE SAME RESULT, IF YOU USE $V = \frac{1}{3} r^2 h$ FOR THE VOLUME OF THE CONE.

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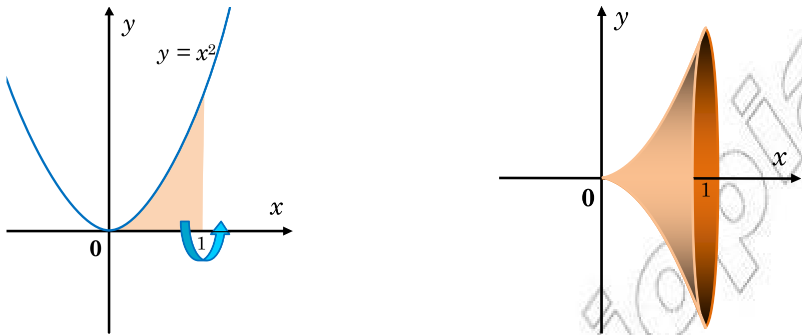
Example 15 FIND THE VOLUME OF THE SOLID GENERATED BY REVOLVING THE REGION BOUNDED BY THE GRAPH OF $y = x^2$ AND THE x-AXIS BETWEEN $x = 0$ AND $x = 1$ ABOUT THE x-AXIS.

Solution



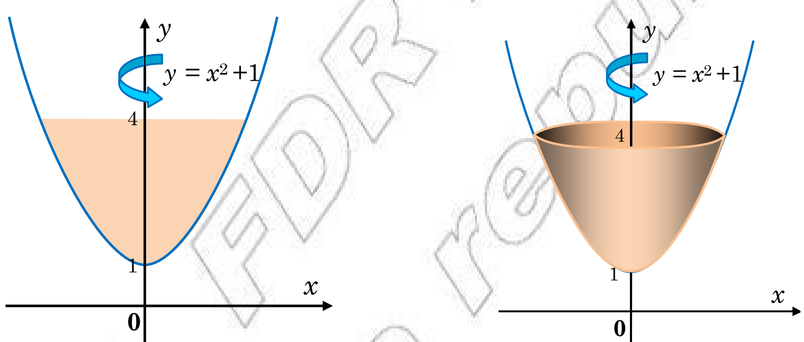
A *The region to be rotated* B *The solid of revolution*

Figure 5.32

$$V = \int_0^3 \left(x^2\right)^2 dx = \frac{x^5}{5}\Bigg|_0^1 = \frac{\ }{5}$$

Example 16 THE AREA BOUNDED BY THE GRAPH OF $y = x^2 + 1$ AND THE LINE $y = 4$ ROTATES ABOUT THE y-AXIS, FIND THE VOLUME OF THE SOLID GENERATED.

Solution



A *The region being rotated about the y – axis.* B *The solid of revolution*

Figure 5.33

$y = x^2 + 1 \Rightarrow x = \pm\sqrt{y-1}$. HERE, YOU HAVE HORIZONTAL CROSS SECTIONS.

$$V = \int_1^4 \left(\sqrt{y-1}^{\,2}\right) dx = \int_1^4 (y-1)\,dy = \left(\frac{y^2}{2} - y\right)\Bigg|_1^4 = \left(\frac{16}{2} - 4\right) - \left(\frac{1}{2} - 1\right) = \frac{9}{2}$$

Example 17 FIND THE VOLUME OF THE SOLID OF REVOLUTION WHEN THE REGION ENCLOSED BY $y = e^x - 1$ AND THE x-AXIS FROM $x = \ln\left(\frac{1}{2}\right)$ TO $x = \ln(2)$ ROTATES ABOUT THE x-AXIS.

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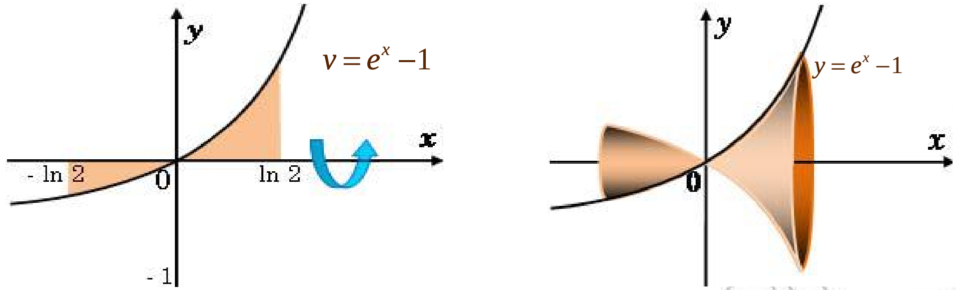
Solution $\ln\left(\frac{1}{2}\right) = -\ln(2)$



A *The region to be rotated* **B** *The solid of revolution*

Figure 5.34

$$V = \int_{-\ln(2)}^{\ln(2)} \left(e^x - 1\right)^2 dx$$

$$= \int_{-\ln(2)}^{\ln(2)} \left(e^{2x} - 2e^x + 1\right) dx = \left(\frac{e^{2x}}{2} - 2e^x + x\right)\Big|_{-\ln 2}^{\ln 2}$$

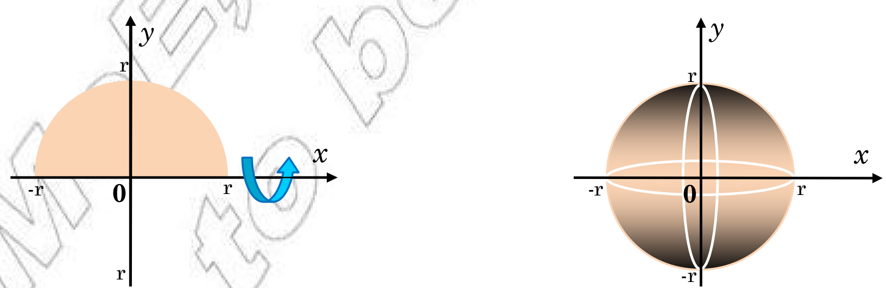
$$= \left(\frac{e^{2\ln 2}}{2} - 2e^{\ln 2} + \ln 2\right) - \left(\frac{e^{-2\ln 2}}{2} - 2e^{-\ln 2} - \ln 2\right)$$

$$= \left(2 - 4 + \ln 2\right) - \left(\frac{1}{8} - 1 - \ln 2\right)$$

$$= \left(-2 + \ln 2\right) - \left(-\frac{7}{8} - \ln 2\right) = \left(-2 + 2\ln 2 + \frac{7}{8}\right) = \left(2\ln 2 - \frac{9}{8}\right)$$

Example 18 Using the volume of a solid of revolution, show that the volume of a sphere of radius r is $\frac{4}{3} r^3$.

Solution In Activity 5.11 you should have seen that a sphere of radius r is generated when the semicircular region $x^2 + y^2 \le r^2$, $0 \le y \le r$ revolves around the x-axis.



A *The semicircular region to be rotated* **B** *A sphere of radius r*

Figure 5.35

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$$x^2 + y^2 = r^2\ ;\ 0 \le y \le r \quad \Rightarrow \quad y = \sqrt{r^2 - x^2}$$

THE VOLUME

$$V = \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx = \int_{-r}^{r} \left(r^2 - x^2\right) dx = \left(r^2 x - \frac{x^3}{3}\right)\Bigg|_{-r}^{r}$$

$$= \pi\left(r^2(r) - \frac{r^3}{3} - \left(r^2(-r) - \frac{(-r)^3}{3}\right)\right) = \pi\left(r^3 - \frac{r^3}{3} - \left(-r^3 + \frac{r^3}{3}\right)\right) = \frac{4}{3}\pi r^3$$

Example 19 FIND THE VOLUME OF WATER IN A SPHERICAL BOWL OF RADIUS 5 M, IF ITS MAXIMUM DEPTH IS 2 M.

Solution FROM FIGURE 5.36, YOU CAN DETERMINE THE RADIUS OF THE SURFACE OF THE WATER WHICH IS 4 M.

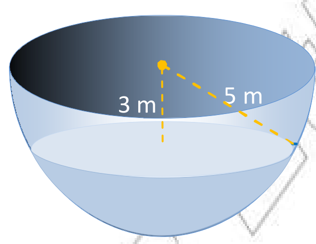

Figure 5.36

THE HEMISPHERE CAN BE GENERATED BY THE QUARTER OF THE CIRCULAR REGION.

$x^2 + y^2 = 25\ ;\ 0 \le x \le 5$ AND $-5 \le y \le 0$ REVOLVING ABOUT THE y-AXIS.

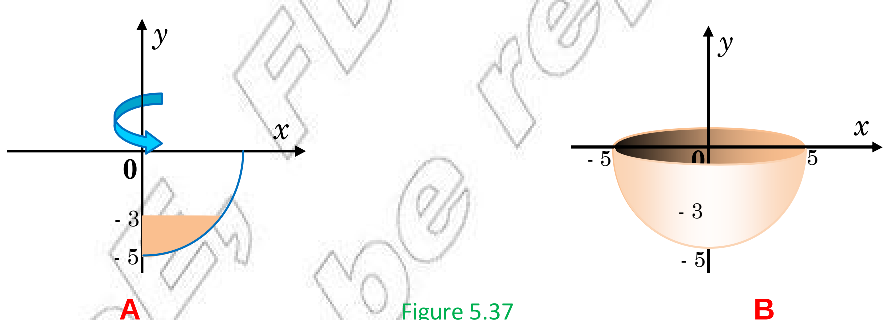

Figure 5.37

THE VOLUME OF THE WATER $= \int_{-5}^{-3} \left(\sqrt{25 - y^2}\right)^2 dy = \int_{-5}^{-3} \left(25 - y^2\right) dy$

$$= \pi\left(25y - \frac{y^3}{3}\right)\Bigg|_{-5}^{-3} = \pi\left(25(-3) - \left(\frac{-27}{3}\right) - \left(-125 + \frac{125}{3}\right)\right)$$

$$= \frac{52}{3} \text{ CM}^3.$$

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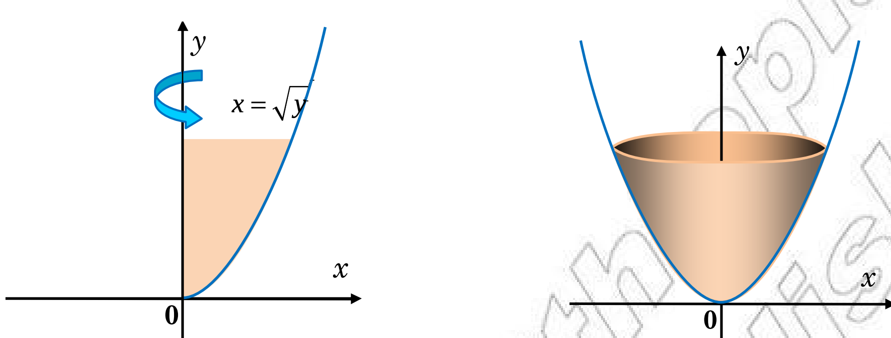
Example 20 FIND THE VOLUME OF THE SOLID OF REVOLUTION GENERATED BY REVOLVING THE REGION ENCLOSED BY THE CURVE $x = \sqrt{y}$ AND THE y-AXIS ABOUT THE y-AXIS FROM $y = 0$ TO $y = 4$.

SOLUTION $V = \int_0^4 \left(\sqrt{y}\right)^2 dy = \int_0^4 y\, dy = \left(\frac{y^2}{2}\right)\Big|_0^4 = 8$



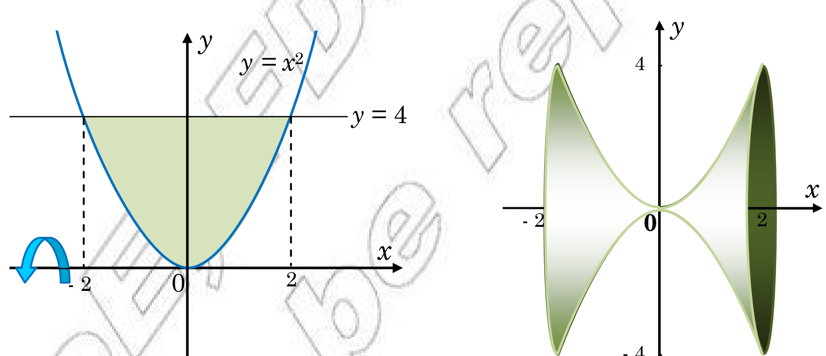
A *The region to rotated about the y – axis* **B** *The solid of revolution*

Figure 5.38

Example 21 IF THE REGION BOUNDED BY THE CURVE $y = x^2$ AND THE LINE $y = 4$ ROTATES ABOUT THE x-AXIS, FIND THE VOLUME OF THE SOLID OF REVOLUTION.

Solution THE FIRST STEP IS TO DETERMINE THE INTERSECTION POINT OF THE LINE AND THE CURVE AND THEN SKETCH BOTH GRAPHS TOGETHER.

$x^2 = 4 \Rightarrow x = \pm 2$



A *The region to be rotated about the x – axis* **B** *The solid of revolution*

Figure 5.39

THE SOLID OF REVOLUTION IS A CYLINDER WITH A VACANT SPACE GENERATED BY THE AREA BOUNDED BY $y = x^2$ AND THE x-AXIS FROM $x = -2$ TO $x = 2$.

LET V_1 BE THE VOLUME OF VACANT SPACE.

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THEN $V_1 = \int_{-2}^{2}\left(x^2\right)^2 dx = \left.\frac{x^5}{5}\right|_{-2}^{2} = \left(\frac{32}{5} - \left(-\frac{32}{5}\right)\right) = \frac{64}{5}$

LET V_2 BE THE VOLUME OF THE CYLINDER, THEN

$$V_2 = \int_{-2}^{2} 4^2 dx = 16 \left. x \right|_{-2}^{2} = 16 \quad (2-(-2)) = 64$$

THUS, THE VOLUME V OF THE REQUIRED SOLID IS

$$V = V_2 - V_1 = 64 - \frac{64}{5} = \frac{256}{5}$$

OBSERVE THAT

$$V = V_2 - V_1 = \int_{-2}^{2} 4^2 dx - \int_{-2}^{2}\left(x^2\right)^2 dx = \int_{-2}^{2}\left(4^2 - \left(x^2\right)^2\right) dx$$

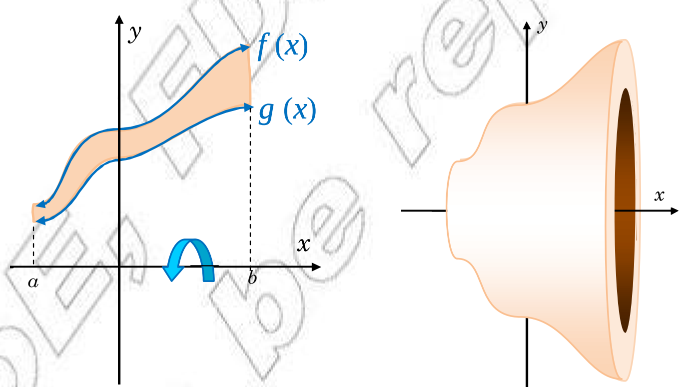
$$= \int_{-2}^{-2}\left(4^2 - x^4\right) dx = \int_{-2}^{-2}\left(16 - x^4\right) dx = \int_{-2}^{2} 16\, dx - \int_{-2}^{2} x^4\, dx$$

$$= \times 16\, x \Big|_{-2}^{2} - \left.\frac{x^5}{5}\right|_{-2}^{2} = (32 + 32) - \left(\frac{32}{5} + \frac{32}{5}\right)$$

$$= 64 - \frac{64}{5} = \frac{256}{5}$$

FROM THE ABOVE OBSERVATION, CAN YOU SEE HOW TO CALCULATE THE VOLUME OF A SOLID OF REVOLUTION GENERATED BY AN AREA ENCLOSED BY TWO CURVES?

CONSIDER THE REGION ENCLOSED BY THE CURVES $y = f(x)$ AND $y = g(x)$ BETWEEN $x = a$ AND $x = b$.



A *The region enclosed by two curves* **B** *The solid of revolution*

Figure 5.40

USING THE CONCEPT SEEN IN EXAMPLE 21, YOU HAVE THE VOLUME V OF THE SOLID OF REVOLUTION TO BE

$$V = \int_{a}^{b}\left(\left(f(x)\right)^2 - \left(g(x)\right)^2\right) dx$$

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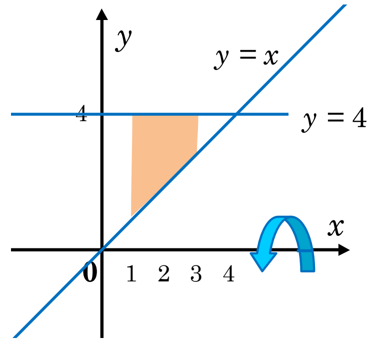
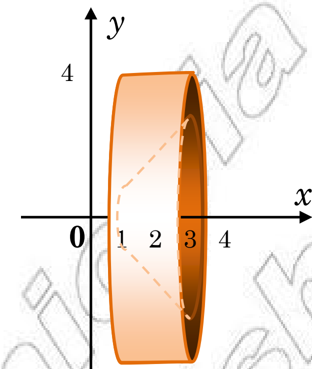
Example 22 FIND THE VOLUME OF SOLID OF REVOLUTION ABOUT THE x-AXIS GENERATED BY REVOLVING THE AREA BETWEEN $y = x$ AND THE LINE $y = 4$ FROM $x = 1$ TO $x = 3$.

Solution



A *The region to be rotated*



B *The solid of revolution*

Figure 5.41

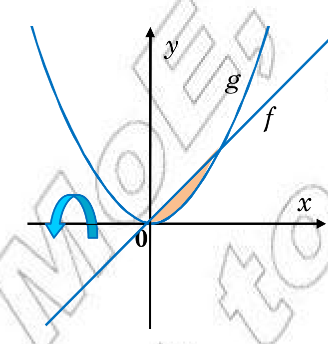
USING THE FORMULA $V = \int_a^b \left((f(x))^2 - (g(x))^2 \right)$; YOU HAVE

$$V = \int_1^3 \left(4^2 - x^2\right) dx = \left(16x - \frac{x^3}{3}\right)\Bigg|_1^3 = (48-9) - \left(16 - \frac{1}{3}\right) = \frac{70}{3}$$

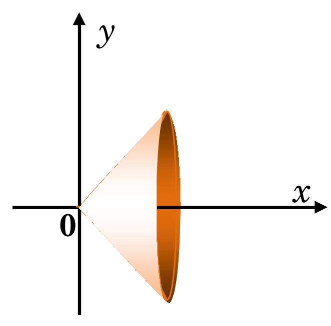
Example 23 IF THE REGION ENCLOSED BY THE GRAPHS OF $f(x) = x$ AND $g(x) = x^2$ FROM $x = 0$ TO $x = 1$ ROTATES ABOUT THE x-AXIS, FIND THE VOLUME OF THE SOLID OF REVOLUTION.

Solution

$$V = \int_0^1 \left(x^2 - \left(x^2\right)^2\right) dx = \left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1 = \left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2}{15}$$



A



B

Figure 5.42

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Example 24 *Work done by a variable force*

THE WORK DONE BY A FORCE F THROUGH A DISPLACEMENT FROM x_1 TO x_2 IS

$$\int_{x_1}^{x_2} |F|\, dx$$

FIND THE WORK DONE WHEN A PARTICLE IS MOVED THROUGH A DISPLACEMENT OF 10M ALONG A SMOOTH HORIZONTAL SURFACE BY A FORCE OF MAGNITUDE $\left(9-\frac{1}{2}x\right)$ N, WHERE x IS THE DISPLACEMENT OF THE PARTICLE FROM ITS INITIAL POSITION, IN METRES.

Solution

$$\text{WORK DONE} = \int_0^{10} |F|\, dx = \int_0^{10}\left(9-\frac{1}{2}x\right)dx = 9x - \frac{x^2}{4}\Big|_0^{10} = 90 - \frac{100}{4} = 65.$$

Motion of a particle in a straight line

SUPPOSE A PARTICLE P MOVES ALONG A STRAIGHT LINE $\overrightarrow{OX}$ WITH O AS ITS INITIAL POINT.

THE VELOCITY v IS THE RATE AT WHICH THE DISPLACEMENT s INCREASES WITH RESPECT TO TIME t.

$$\Rightarrow v = \frac{ds}{dt} \Rightarrow \int v\, dt = \int ds \qquad \Rightarrow s = \int v\, dt$$

THE ACCELERATION a IS THE RATE AT WHICH THE VELOCITY INCREASES WITH RESPECT TO TIME t.

$$\Rightarrow a = \frac{dv}{dt} \Rightarrow \int a\, dt = \int dv \Rightarrow v = \int a\, dt$$

Example 25 SUPPOSE A PARTICLE P MOVES ALONG A STRAIGHT LINE $\overrightarrow{OX}$ WITH AN ACCELERATION OF $3.5t$. WHEN t = 2 SEC, P HAS A DISPLACEMENT OF 10 M FROM O AND A VELOCITY OF 15 M/SEC. FIND THE VELOCITY v AND THE DISPLACEMENT s WHEN t = 5 SEC.

Solution USING THE GIVEN INFORMATION YOU HAVE,

$$v = \int a\, dt = \int 3.5t\, dt = \frac{3.5}{2}t^2 + c. \text{ BUT } v(2) = 15 \Rightarrow 15 = \frac{3.5}{2}(2)^2 + c \Rightarrow c = 8.$$

$$v = \frac{7}{4}t^2 + 8.$$

$$\text{ALSO, } s = \int v\,dt \Rightarrow s = \int\left(\frac{7}{4}t^2 + 8\right)dt = \frac{7}{12}t^3 + 8t + c$$

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BUT $s(2) = 10 \Rightarrow 10 = \frac{7}{12}(2)^3 + 8(2) + c$

$$\Rightarrow c = -\frac{32}{3} \quad \Rightarrow s = \frac{7}{12}t^3 + 8t - \frac{32}{3}$$

THEREFORE, WHEN $t = 5$,

a THE VELOCITY, $v = \frac{7}{4}(5)^2 + 8 = 51.75$ M/SEC

b THE DISPLACEMENT, $s = \frac{7}{12}(5)^3 + 8(5) - \frac{32}{3} = 102.25$ M

Exercise 5.12

1 FIND THE VOLUME OF THE SOLID OF REVOLUTION GENERATED ABOUT THE x-AXIS BY REVOLVING THE REGION ENCLOSED BY THE GIVEN FUNCTION, THE x-AXIS AND THE VERTICAL LINES.

A $y = 2x$; $x = 0$ AND $x = 1$ **B** $y = x^2 + 1$; $x = -1$ AND $x = 2$

C $y = e^x$; $x = 1$ AND $x = 2$ **D** $y = \sin x$; $x = \frac{\pi}{3}$ AND $x = \frac{\pi}{2}$

E $y = |x|$; $x = -3$ AND $x = 1$ **F** $y = 2^x$; $x = -2$ AND $x = 3$

G $y = x^3$; $x = -1$ AND $x = 2$

2 FIND THE VOLUME OF THE SOLID OF REVOLUTION GENERATED ABOUT THE x-AXIS BY REVOLVING THE REGION ENCLOSED BY THE GRAPHS OF THE GIVEN FUNCTIONS.

A $f(x) = 4x - x^2$ AND $g(x) = 3$

B $f(x) = x^3$ AND $g(x) = x$

C $f(x) = \sin x$ AND $g(x) = \cos x$ FROM $x = 0$ TO $x = \frac{\pi}{2}$

D $f(x) = x^2$, $g(x) = |x|$ FROM $x = -2$ TO $x = 2$

3 USING THE VOLUME OF REVOLUTION, PROVE THAT THE VOLUME OF A FRUSTUM OF A RIGHT CIRCULAR CONE OF RADII r AND R AND HEIGHT h IS $\frac{1}{3}\pi h\left(R^2 + rR + r^2\right)$.

4 A PARTICLE P STARTS FROM A POINT A WITH VELOCITY 4 M/SEC. IF IT IS MOVING ALONG A STRAIGHT LINE AB WITH AN ACCELERATION $t^2 + 1.5$ AT TIME t SECONDS, FIND

A THE ACCELERATION **B** THE VELOCITY AND

C THE DISPLACEMENT AFTER TEN SECONDS

Key Terms

acceleration	differentiation	partial fraction
anti derivative	displacement	substitution
area	fundamental theorem	velocity
by parts	indefinite integral	volume of revolution
definite integral	integration	work done by force

Summary

1 *Anti derivative or Indefinite integral*

LET $f(x)$ BE A FUNCTION, THEN

- ✓ $F(x)$ IS SAID TO BE AN ANTIDERIVATIVE OF $f(x)$ IFF $F'(x) = f(x)$.
- ✓ THE SET OF ANTIDERIVATIVES OF $f(x)$ IS SAID TO BE THE INDEFINITE INTEGRAL OF $f(x)$.
- ✓ THE INDEFINITE INTEGRAL OF $f(x)$ IS DENOTED BY $\int f(x)\, dx$
- ✓ IF $F(x)$ AND $G(x)$ ARE ANTI DERIVATIVES OF $f(x)$, THEN THE DIFFERENCE BETWEEN $F(x)$ AND $G(x)$ IS A CONSTANT.

2 *The Integral of Some Functions*

The Integral of power functions

I $\int x^r\, dx = \frac{x^{r+1}}{r+1} + c;\ r \neq -1.$

II IF $r = -1$, THEN $\int \frac{1}{x}\, dx = \ln|x| + c$.

III $\int kx^r\, dx = k\int x^r\, dx$

The Integral of trigonometric functions

I $\int \cos x\, dx = \sin x + c$

II $\int \sin x\, dx = -\cos x + c$

III $\int \sec^2 x\, dx = \tan x + c$

IV $\int \sec x \tan x\, dx = \sec x + c$

V $\int \tan x\, dx = -\ln|\cos x| + c$

VI $\int \csc x \cot x\, dx = -\csc x + c$

VII $\int \csc^2 x\, dx = -\cot x + c$

The Integral of exponential functions

I $\int e^x\, dx = e^x + c$

II $\int a^x\, dx = \frac{a^x}{\ln a} + c;\ a > 0$ AND $a \neq 1$

The integral of logarithmic functions

I $\int \ln x\, dx = x \ln x - x + c$

II $\int \log_a x\, dx = \frac{1}{\ln a}\left(x \ln x - x\right) + c$

3 *The Integral of a sum or difference of functions*

I $\int kf(x)dx = k\int f(x)\, dx$

II $\int \left(f(x) \pm g(x)dx\right) = \int f(x)dx \pm \int g(x)dx$

4 Techniques of Integration

Integration by substitution

$\int f\left(g(x)\right).g'(x)\,dx = \int f(u)\,du$; WHERE $u = g(x)$.

I $\int f'(x) f(x) dx = \frac{\left(f(x)\right)^2}{2} + c$ II $\int \frac{f'(x)}{f(x)} dx = \ln\left|f(x)\right| + c$

Integration by parts

$\int u \frac{d}{dx} = uv - \int v \frac{du}{dx}$

5 Fundamental Theorem of Calculus

IF $f(x) = F'(x)$, THEN $\int_a^b f(x)\,dx = F(b) - F(a)$

6 Properties of definite integrals

I $\int_a^b k\, f(x)\,dx = k\int_a^b f(x)\,dx$

II $\int_a^b \left(f(x) \pm g(x)\right) dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

III IF $f(x) \geq 0$ ON $[a, b]$, THEN $\int_a^b f(x)\,dx \geq 0$

IV $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

V $\int_a^a f(x)\,dx = 0$

VI $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$; $a \leq c < b$.

VII IF $u = g(x)$, $\int_a^b f\left(g(x)\right).g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

7 Applications of the definite integral

I THE AREA A BOUNDED BY TWO CONTINUOUS CURVES $y = f(x)$ AND $y = g(x)$ ON $[a, b]$ WITH $f(x) \geq g(x)$ $\forall x \in [a, b]$ IS

$$A = \int_a^b \left(f(x) - g(x)\right) dx.$$

II THE VOLUME V OF A SOLID OF REVOLUTION GENERATED BY REVOLVING THE REGION BOUNDED BY $y = f(x)$ AND $y = g(x)$ WITH $f(x) \geq g(x)$ $\forall x \in [a,b]$ ABOUT THE x-AXIS IS

$$V = \int_a^b \left(\left(f(x)\right)^2 - \left(g(x)\right)^2\right) dx.$$

Review Exercises on Unit 5

IN EXERCISES 1 – 60, INTEGRATE THE EXPRESSION WITH RESPECT TO x.

1	$\frac{1}{2}x$	2	$2x+5$	3	x^2-3x+2	4	$3x^5$
5	x^7	6	$x^{\frac{3}{2}}$	7	$x^{\frac{1}{3}}$	8	$x^{\frac{1}{3}}+x\sqrt{x}-\frac{1}{2x}+x$
9	$2x^{-3}$	10	$x^{-\frac{2}{5}}$	11	$\sin(x+2)$	12	$4^x-\sin x$
13	$\tan(3x-4)$	14	$x(x^2+1)$	15	$\sqrt{2x+7}$	16	$(3x+5)^{13}$
17	$(x^2-1)^3$	18	$(x+1)(x^2+2x+5)^{10}$	19	$\frac{x}{x+1}$	20	$\frac{1}{x^2-16}$
21	$x\sqrt{(x^2+4)^5}$	22	$\frac{x}{x^2-2x-3}$	23	2^{4X+3}	24	$x\log\sqrt{x^2+1}$
25	$\sin^{(n)}(x)\cos x$	26	$\frac{3^{x+1}}{5^{1-4x}}$	27	$\frac{\ln x}{x}$	28	$x\sin(3x^2)$
29	$x\lvert x\rvert$	30	$\sqrt{6+x}$	31	$\frac{\sqrt{x}+x}{x\sqrt[3]{x}}$	32	$(1+2^x)^2$
33	$\frac{2\sqrt{x}}{\sqrt{x}}$	34	$\frac{2x+1}{4^{x^2+x+1}}$	35	$2^x\,2x\sqrt{1+2^x}$	36	$\frac{(E^{x+3})(3^{x+5})}{2^{3X-2}}$
37	$\frac{\cos x}{3+\sin x}$	38	$\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$	39	$\frac{\sec^2\sqrt{x}}{\sqrt{x}}$	40	$\frac{\sin x}{\cos^7 x}$
41	xe^{x^2}	42	$x^{-2}e^{\frac{1}{x}}$	43	$\frac{e^{\frac{1}{x}}}{x^2}$	44	$\frac{x+1}{x^2+2x+4}$
45	$\frac{4}{(x+3)^2}$	46	$\frac{x^2}{x+3}$	47	$(2x+1)(x^2+x+3)^{10}$	48	$x\sqrt{9+x}^{3}$
49	$\cos x\, e^{\sin x}$	50	$\frac{x}{\sqrt{x^2+5x}}$	51	$(1+2e^x)^2$	52	$\sin^3\left(\frac{x}{3}\right)$
53	$\frac{3x}{x^2-1}$	54	$\frac{3x^2}{x^2-9}$	55	$\frac{x}{(x-2)(x+1)^2}$	56	$\frac{3x+2}{(x+3)^2}$
57	$\frac{4}{x^2(x+1)^2}$	58	$\frac{2x^2+1}{(x+1)^2(x+3)}$	59	$\frac{x^3+1}{x^2(x-4)}$	60	$\frac{x}{(x^2-1)(x+3)}$

IN EXERCISES 61 – 85 EVALUATE THE DEFINITE INTEGRAL.

61	$\int_a^b dx$	**62**	$\int_{e-1}^{e+1} 4dx$	**63**	$\int_2^3 (x-5)\,dx$
64	$\int_1^2 6x^3 dx$	**65**	$\int_0^1 e^x dx$	**66**	$\int_1^4 \sqrt{x}\,dx$
67	$\int_{\sqrt{2}}^3 3^x dx$	**68**	$\int_1^8 x^{\frac{1}{3}}\,dx$	**69**	$\int_1^3 \sqrt{x}\left(1-\frac{1}{x}\right)dx$
70	$\int_{-1}^1 e^{x+3} dx$	**71**	$\int_0^1 3^{2x+5} dx$	**72**	$\int_{\frac{1}{2}}^1 2^{3x-2} dx$
73	$\int_0^1 \frac{1}{x+1}dx$	**74**	$\int_{-2}^2 \left(e^x+e^{-x}\right)dx$	**75**	$\int_{\frac{1}{n}}^1 e^{nx} dx$
76	$\int_2^3 \frac{x}{x+5}dx$	**77**	$\int_0^3 x\sqrt{x^2+1}\,dx$	**78**	$\int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$
79	$\int_{\frac{}{2}} \cos(5x)\,dx$	**80**	$\int_{\frac{}{3}}^{\frac{}{2}} \sin x \cos^3 x\,dx$	**81**	$\int_0^1 \frac{t}{4^{t^2-1}}dt$
82	$\int_0 \frac{\sin x}{4+\cos x}dx$	**83**	$\int_{-2}^1 (x+1)\sqrt{x+2}\,dx$	**84**	$\int_1^0 x\left(8x^2-1\right)^6 dx$
85	$\int_1^2 \frac{2x-3}{\left(x^2-3x+1\right)}dx$				

IN EXERCISES 86 – 97 FIND THE AREA OF THE REGION BOUNDED BY THE GRAPH OF f, THE x-AXIS AND THE LINES $x = a$ AND $x = b$.

86	$f(x) = 4;\ a = -1,\ b = 2$	**87**	$f(x) = 3x;\ a = -3,\ b = -1$
88	$f(x) = 3x + 1;\ a = 0,\ b = 3$	**89**	$f(x) = 2x^2 + 1;\ a = 0,\ b = 3$
90	$f(x) = 1 - 4x^2;\ a = -1,\ b = 1$	**91**	$f(x) = x^3;\ a = -\frac{1}{2},\ b = 2$
92	$f(x) = e^x;\ a = -1,\ b = 4$	**93**	$f(x) = \frac{x}{x+1};\ a = -\frac{1}{2},\ b = 3$
94	$f(x) = \sqrt{x} + \frac{1}{\sqrt{x}};\ a = \frac{1}{4},\ b = 4$	**95**	$f(x) = \ln x;\ a = \frac{1}{e},\ b = e$
96	$f(x) = x^3 - 2x^2 - 5x + 6;\ a = -2,\ b = 3$	**97**	$f(x) = \lvert x^2 - 1 \rvert;\ a = -3,\ b = 2$

98 FIND EACH OF THE FOLLOWING SHADED AREAS.

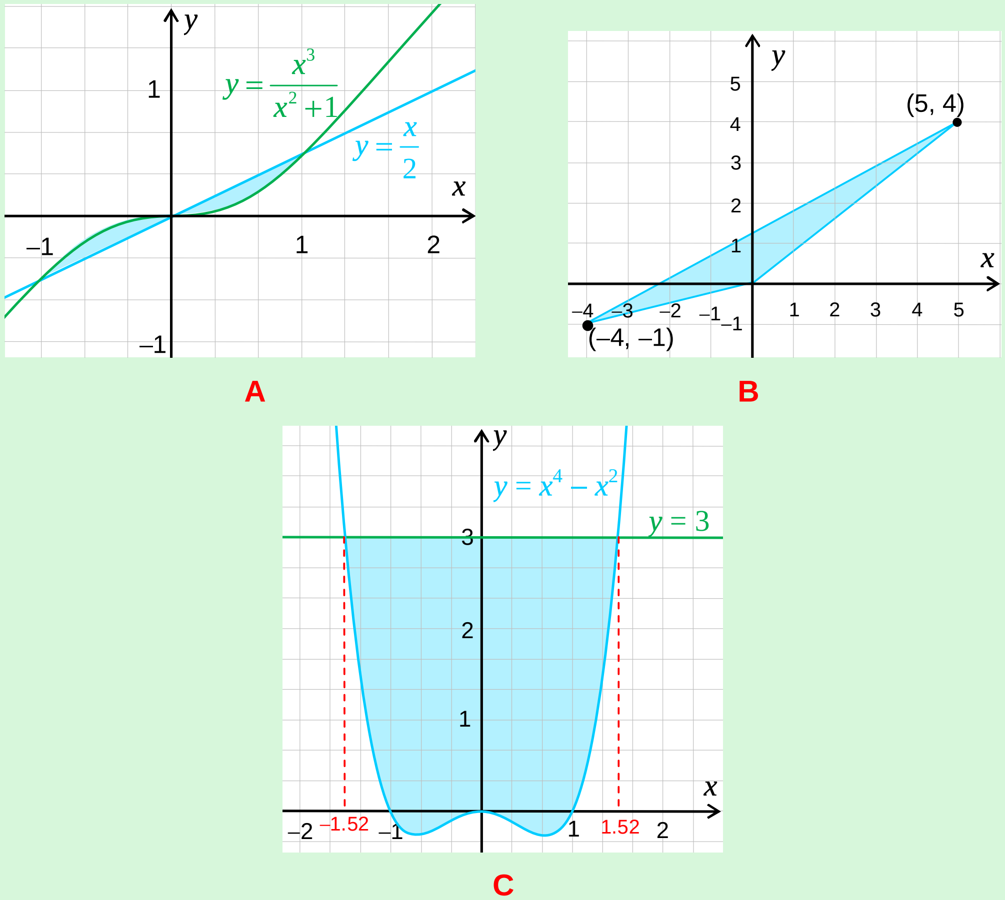

Figure 5.43

99 FIND THE AREA OF THE REGION ENCLOSED BY

A $f(x) = x$, $y = \frac{1}{x}$ AND $y = 4$ **B** $f(x) = 4 - x^2$ AND $g(x) = 3x$.

100 FIND THE VOLUME OF THE SOLID GENERATED WHEN THE REGION ENCLOSED BY THE x-AXIS AND THE GIVEN CURVES AND LINES IS ROTATED ABOUT THE x-AXIS.

A $y = 4x - x^2$ **B** $y = x^3 + 1$; $x = -1$, $x = 2$

101 FIND THE VOLUME OF THE SOLID GENERATED WHEN THE REGION BOUNDED BY $y = 3x$ AND $y = x^2 + 2$ ROTATES ABOUT THE x-AXIS.

102 FIND THE VOLUME OF THE SOLID OF REVOLUTION GENERATED WHEN THE REGION ENCLOS CURVE $y = e^x$, THE y-AXIS AND THE LINE $y = e$ ROTATES ABOUT THE y-AXIS.